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The geometry of log-symplectic manifolds

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Preface

Symplectic geometry arose as the mathematical framework to describe classical mechanics. An extension of symplectic geometry is provided by Poisson geometry, and indeed symplectic manifolds are exactly the non-degenerate Poisson manifolds. Generic Poisson manifolds are singular objects by nature, and this makes their geometry highly nontrivial and rather wild. It is therefore desirable to consider Poisson manifolds with well-controlled singularities as more tractable working examples.

In this thesis, we consider such a class of Poisson manifolds, that in (part of) the literature is referred to as "log-symplectic manifolds". They form a convenient intermediate level between the symplectic world and the generic Poisson world. A log-symplectic structure degenerates in a mild fashion along a hypersurface, called its singular locus, but it is non-degenerate (i.e. symplectic) elsewhere. So these structures do not stray too far from being symplectic, and their behaviour is indeed analogous to that of symplectic structures in many respects. Their geometry is nontrivial but accessible, and as such log-symplectic structures have become a topic of intense research during the past 5-6 years.

Broadly speaking, the main question we address in this thesis is the following: "What does a log-symplectic structure look like near its singular locus?". Of particular interest to us are normal form theorems that give model answers to this question. The general theory on log-symplectic structures that we present should be considered as background material and as a necessary tool to build towards theorems that answer our core question.

I thank my supervisor, prof. Marco Zambon, for his help and guidance and for providing a comfortable working climate. I also greatly benefited from the Poisson Geometry Learning Seminar, organized by Marco Zambon and Ori Yudilevich during the past academic year. It enabled me to learn from scratch the basics of Poisson geometry in a sensible way.

Summary

Log-symplectic manifolds, which are the objects under consideration in this thesis, form a natural generalization of symplectic manifolds that arises in Poisson geometry. We start by recalling the basics of symplectic geometry in Chapter 1, with emphasis on the Darboux-Moser theorems. In Chapter 2, we give an introduction to Poisson geometry. Since the author was not familiar with Poisson geometry prior to writing this thesis, the exposition in Chapter 2 is rather extensive and detailed.

Having established the needed preliminaries, we introduce log-symplectic structures in Chapter 3. A Poisson bivector Π on a manifold M^{2n} is called log-symplectic if the top wedge power $\wedge^n\Pi$ is transverse to the zero section of the line bundle $\wedge^{2n}TM$. The zero set $Z = (\wedge^n\Pi)^{-1}(0)$ where the bivector Π degenerates turns out to be a smooth hypersurface, which we call the singular locus of Π. The first main statement of the thesis is Theorem 3.2.2, which appears in [GMP2]. It gives the local model for a log-symplectic structure Π around a point in its singular locus Z, namely

$$\Pi = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i},$$

where Z is locally defined by $y_1 = 0$. As a consequence of this normal form, we obtain that the singular locus Z is a Poisson submanifold, with an induced corank-one Poisson structure.

Log-symplectic structures are described conveniently in the language of b-geometry. This formalism addresses b-manifolds, which are pairs (M, Z) consisting of a manifold M and a distinguished hypersurface $Z \subset M$. Following [GMP2] and [MO], Chapter 4 introduces the basic concepts regarding b-geometry. We fill in some details and proofs that are not given in the literature. The key result is Theorem 4.2.10, which establishes that log-symplectic structures are in fact the symplectic structures of the b-category. This point of view allows us to use symplectic techniques in the study of log-symplectic manifolds. Of particular importance is the Moser Theorem 4.2.7. We also obtain cohomological obstructions to the existence of a log-symplectic structure, similar to those in symplectic geometry.

Chapter 5 describes log-symplectic structures semilocally, in a neighborhood of the singular locus. As such, it is the most important chapter. The second main statement of this thesis is Theorem 5.2.1, which appeared in [BOT]. It gives a normal form for orientable log-symplectic structures (M, Z, Π), valid in a tubular neighborhood U of the singular locus Z:

$$\Pi|_U = X_Z \wedge t\frac{\partial}{\partial t} + \Pi_Z,$$

where X_Z is the restriction to Z of a modular vector field on M and Π_Z is the restriction of Π to $Z \leftrightarrow \{t = 0\}$. The second half of Chapter 5 is dedicated to log-symplectic extensions of corank-one Poisson structures. Following [GMP2], Theorem 5.3.1 determines when a given corank-one Poisson manifold (Z, Π_Z) arises as the singular locus of a log-symplectic structure. Next, we ask ourselves to what extent such a log-symplectic extension is unique. We answer this question in Subsection 5.3.2, most of which is our own work: the material in question is also addressed

in [GMP2], but some parts of the exposition given there need fixing (see Remark 5.3.12). As such, not all statements we present in Subsection 5.3.2 are original, but most of the proofs are. In particular, we prove the third main result of the thesis in Theorem 5.3.13, which states that, up to an appropriate notion of equivalence, the log-symplectic extensions of a corank-one Poisson structure (Z, Π_Z), defined on some tubular neighborhood of Z, are parametrized by the cohomology classes in $H^1_{\Pi_Z}(Z)$ of Poisson vector fields on Z that are transverse to the symplectic leaves.

In Chapter 6, we give a description of compact corank-one Poisson manifolds endowed with a closed one-form defining the symplectic foliation, and a closed two-form extending the symplectic form on each leaf. Following [GMP1], we define two foliation invariants and we show that such Poisson manifolds are in fact mapping tori, whence fibrations over the circle S^1. The results in this chapter apply in particular to the singular locus of a log-symplectic structure.

At last, in Chapter 7, we scratch the surface of some aspects of log-symplectic structures that were not treated in detail in this thesis.

Contents

Chapter 1

Preliminary Symplectic Geometry

Symplectic geometry is a branch in differential geometry that studies symplectic manifolds. It arose as the mathematical framework to describe classical mechanics. Nowadays, it is an independent field of study, significantly stimulated by interactions with mathematical physics and topology, amongst others.

In this thesis, we study an extension of symplectic manifolds, called log-symplectic manifolds. Their behaviour is in many respects similar to that of symplectic manifolds, and many results about symplectic structures can be generalized to the log-symplectic setting. In this preliminary chapter, we recall some of the main concepts in symplectic geometry. It will be interesting to see how these relate to their log-symplectic analogs.

This chapter is based on the lectures of the course "Symplectic Geometry" taught at KU Leuven. Most of what is written below can also be found in [Ca].

1.1 Symplectic vector spaces

Definition 1.1.1. A symplectic vector space is a pair (V, Ω), where V is a real, finite dimensional vector space and $\Omega : V \times V \to \mathbb{R}$ is a bilinear form which is:

(i) Skew-symmetric: $\Omega(v, w) = -\Omega(w, v)$ for all $v, w \in V$.

(ii) Non-degenerate: $\mathrm{Ker}(\Omega) := \{v \in V : \Omega(v, w) = 0 \ \forall w \in V\} = \{0\}$.

Remark 1.1.2. With a bilinear form Ω comes a linear map

$$\widetilde{\Omega} : V \to V^* : v \mapsto \Omega(\cdot, v).$$

Non-degeneracy of Ω is equivalent with $\widetilde{\Omega}$ being an isomorphism. Now let $\beta = \{v_1, \ldots, v_n\}$ be a basis of V, and let $\beta^* = \{v_1^*, \ldots, v_n^*\}$ be the dual basis of V^*. Then the matrix of $\widetilde{\Omega}$ with respect to the bases β and β^* coincides with the matrix of Ω with respect to the basis β. Indeed, denoting $B = [\widetilde{\Omega}]_{\beta}^{\beta^*}$, we have

$$\widetilde{\Omega}(v_j) = \sum_{k=1}^{n} B_{k,j} \alpha_k,$$

which implies

$$\Omega(v_i, v_j) = \widetilde{\Omega}(v_j)(v_i) = B_{i,j}.$$

Hence, to see if Ω is non-degenerate, one only needs to check if its matrix is invertible.

Example 1.1.3. On $\mathbb{R}^{2n}$, denote the canonical basis by $\{e_1, f_1, \dots, e_n, f_n\}$. We define the canonical bilinear form Ω_{can} by the rules

$$\begin{cases} \Omega_{\text{can}}(e_i, e_j) = 0 \\ \Omega_{\text{can}}(f_i, f_j) = 0 \\ \Omega_{\text{can}}(e_i, f_j) = \delta_{i,j} \end{cases} \quad \text{for all } i, j \in \{1, \dots, n\},$$

also imposing bilinearity and skew-symmetry. Note that the matrix of Ω_{can} with respect to this basis is

$$[\Omega_{\text{can}}] = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}. \tag{1.1}$$

Since this matrix is invertible, it follows that Ω_{can} is non-degenerate. Hence $(\mathbb{R}^{2n}, \Omega_{\text{can}})$ is a symplectic vector space.

Definition 1.1.4. A symplectomorphism between symplectic vector spaces (V_1, Ω_1) and (V_2, Ω_2) is an isomorphism of vector spaces $f : V_1 \to V_2$ such that $f^*\Omega_2 = \Omega_1$. Here f^* is the pullback, defined as

$$\big(f^*\Omega_2\big)(v, w) = \Omega_2\big(f(v), f(w)\big) \text{ for all } v, w \in V_1.$$

Example 1.1.3 is prototypical:

Proposition 1.1.5. *Let (V, Ω) be a symplectic vector space. Then $\dim(V) = 2n$ for some $n \in \mathbb{N}$, and (V, Ω) is symplectomorphic to $(\mathbb{R}^{2n}, \Omega_{can})$.*

Proof. Firstly, the standard form theorem for skew-symmetric bilinear maps (Proposition 8.1.1 in the appendix) implies that the rank of Ω is even. Since by assumption, $\text{rank}(\Omega) = \dim(V)$, we get that $\dim(V) = 2n$ for some $n \in \mathbb{N}$. Moreover, the standard form theorem gives a basis $\{v_1, w_1, \dots, v_n, w_n\}$ of V with respect to which the matrix of Ω has the form (1.1). Hence, the map

$$f : (V, \Omega) \to (\mathbb{R}^{2n}, \Omega_{\text{can}}) : \begin{cases} v_i \mapsto e_i \\ w_i \mapsto f_i \end{cases} \quad \text{for } i = 1, \dots, n$$

is the desired symplectomorphism. □

1.2 Symplectic manifolds

Definition 1.2.1. A symplectic manifold is a pair (M, ω) where M is a smooth manifold and $\omega \in \Omega^2(M)$ is a 2-form such that:

(i) ω is closed, i.e. $d\omega = 0$.

(ii) $\omega_p : T_pM \times T_pM \to \mathbb{R}$ is non-degenerate, for all $p \in M$.

Remark 1.2.2. A two-form ω on M is completely determined by its associated vector bundle map $\omega^\flat : TM \to T^*M$, that on the level of sections is given by contraction of ω:

$$\omega^\flat : \mathfrak{X}(M) \to \Omega^1(M) : X \mapsto \iota_X\omega.$$

Non-degeneracy of ω is equivalent with $\omega^\flat$ being a linear isomorphism in the fibers, that is, $\omega^\flat$ being a vector bundle isomorphism.

Example 1.2.3. Consider $\mathbb{R}^{2n}$ with canonical basis $\{e_1, f_1, \ldots, e_n, f_n\}$ and induced coordinates $(q_1, p_1, \ldots, q_n, p_n)$. Then $\omega = \sum_{i=1}^n dq_i \wedge dp_i$ is a symplectic form. Indeed, at any point $x \in M$ we have a basis

$$\left\{ \frac{\partial}{\partial q_1}\bigg|_x, \frac{\partial}{\partial p_1}\bigg|_x, \ldots, \frac{\partial}{\partial q_n}\bigg|_x, \frac{\partial}{\partial p_n}\bigg|_x \right\}$$

of $T_x\mathbb{R}^{2n}$, and under the isomorphism

$$T_x\mathbb{R}^{2n} \to \mathbb{R}^{2n} : \begin{cases} \frac{\partial}{\partial q_i}\Big|_x & \mapsto e_i \\ \frac{\partial}{\partial p_i}\Big|_x & \mapsto f_i \end{cases},$$

ω_x corresponds with Ω_{can}. As Ω_{can} is non-degenerate, so is ω_x. Closedness of ω is clear.

The Darboux Theorem, which will be proved later, says that Example 1.2.3 is the local model for all symplectic manifolds.

Example 1.2.4. Let M be an orientable surface. Then any volume form ω on M is a symplectic form. Indeed, closedness is automatic since M is 2-dimensional:

$$(d\omega)_p \in \wedge^3 T_p^* M = \{0\}.$$

Non-degeneracy is argued for as follows. Let $p \in M$ and choose a basis $\{v_1, v_2\}$ of T_pM. Since ω_p is nonzero, we have that $\omega_p(v_1, v_2) \neq 0$. Now assume that $v \in T_pM$ is such that $\omega_p(v, w) = 0$ for all $w \in T_pM$. Writing $v = \lambda_1 v_1 + \lambda_2 v_2$, we get in particular

$$\begin{cases} 0 = \omega_p(v, v_1) = -\lambda_2 \omega_p(v_1, v_2) \\ 0 = \omega_p(v, v_2) = \lambda_1 \omega_p(v_1, v_2) \end{cases}.$$

As $\omega_p(v_1, v_2) \neq 0$, this implies that $\lambda_1 = \lambda_2 = 0$, hence $v = 0$.

Example 1.2.5. If Q is any manifold, then its cotangent bundle T^*Q is symplectic in a canonical way. Denote by $\pi : T^*Q \to Q : f \in T_x^*Q \mapsto x$ the bundle projection. The tautological one-form $\theta \in \Omega^1(T^*Q)$ is defined by

$$\theta_\xi(v) = \langle \xi, \pi_* v \rangle \qquad \text{for } \xi \in T^*_{\pi(\xi)}Q \quad \text{and} \quad v \in T_\xi(T^*Q),$$

where $\langle \cdot, \cdot \rangle$ is the pairing between $T_{\pi(\xi)}Q$ and $T^*_{\pi(\xi)}Q$. We now coordinatize T^*Q as follows. Choosing local coordinates $(U, q_1, \ldots, q_n)$ on Q defines a local frame $\{dq_1, \ldots, dq_n\}$ of sections for T^*Q. For all $x \in U$, we get a basis $\{d_x q_1, \ldots, d_x q_n\}$ of T_x^*Q and the coordinates it induces on T_x^*Q will be called $(p_1, \ldots, p_n)$. We thus obtain coordinates $(\pi^* q_1, \ldots, \pi^* q_n, p_1, \ldots, p_n)$ on $T^*Q|_U$. We keep writing q_i instead of $\pi^* q_i$. In these coordinates, we have

$$\theta = \sum_{i=1}^n p_i dq_i.$$

Indeed, since $\pi : T^*Q \to Q : (q, p) \mapsto q$, we have

$$\pi_* = \frac{\partial(q_1, \ldots, q_n)}{\partial(q_1, \ldots, q_n, p_1, \ldots, p_n)} = [I_{n\times n} \mid 0_{n\times n}],$$

which gives

$$\theta_{(q,p)} \left(\sum_{i=1}^n a_i \frac{\partial}{\partial q_i} + b_i \frac{\partial}{\partial p_i} \right) = \left\langle \sum_{i=1}^n p_i dq_i, \pi_* \left(\sum_{i=1}^n a_i \frac{\partial}{\partial q_i} + b_i \frac{\partial}{\partial p_i} \right) \right\rangle$$

$$= \left\langle \sum_{i=1}^n p_i dq_i, \sum_{i=1}^n a_i \frac{\partial}{\partial q_i} \right\rangle$$
$$= \sum_{i=1}^n p_i a_i$$
$$= \left(\sum_{i=1}^n p_i dq_i \right) \left(\sum_{i=1}^n a_i \frac{\partial}{\partial q_i} + b_i \frac{\partial}{\partial p_i} \right).$$

We now define $\omega := -d\theta$. It is an exact, hence closed 2-form. And ω is non-degenerate: in coordinates it is given by

$$\omega = -d \left(\sum_{i=1}^n p_i dq_i \right) = \sum_{i=1}^n dq_i \wedge dp_i,$$

and this form is non-degenerate by the same argument as in Example 1.2.3.

Definition 1.2.6. A symplectomorphism between symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is a diffeomorphism $f : M_1 \to M_2$ that satisfies $f^*\omega_2 = \omega_1$.

Not all manifolds are symplectic. We now present some obstructions to the existence of a symplectic structure.

Proposition 1.2.7. *Let (M, ω) be a symplectic manifold. Then* $\dim(M)$ *is even and M is orientable.*

Proof. We have for all $p \in M$ that (T_pM, ω_p) is a symplectic vector space. By Proposition 1.1.5, we get that $\dim(M) = \dim(T_pM)$ is even. Let $\dim(M) = 2n$. As for the orientability of M, we just have to note that ω^n is a volume form on M. Indeed, ω^n is nowhere vanishing by Lemma 8.1.2 in the appendix. $\square$

The next proposition gives cohomological obstructions to the existence of a symplectic structure. Recall that for all $l \in \{0, \dots, \dim(M)\}$, the l-th de Rham cohomology group is defined as

$$H^l(M) = \frac{\{\text{closed } l\text{-forms}\}}{\{\text{exact } l\text{-forms}\}} = \frac{\{\beta \in \Omega^l(M) : d\beta = 0\}}{\{\beta \in \Omega^l(M) : \exists \alpha \in \Omega^{l-1}(M) : \beta = d\alpha\}}.$$

Proposition 1.2.8. *Let (M^{2n}, ω) be a compact symplectic manifold. Then for all $k \in \{1, \dots, n\}$, the de Rham cohomology class $[\omega^k] \in H^{2k}(M)$ is nonzero.*

Proof. First assume by contradiction that $[\omega^n] \in H^{2n}(M)$ is zero. Then ω^n is exact, i.e. there exists $\alpha \in \Omega^{2n-1}(M)$ such that $\omega^n = d\alpha$. Making essential use of compactness of M and Stokes' theorem, we get

$$0 \neq \mathrm{Vol}(M) = \frac{1}{n!} \int_M \omega^n = \frac{1}{n!} \int_M d\alpha = \frac{1}{n!} \int_{\partial M} \alpha = 0,$$

where the last equality holds since $\partial M = \emptyset$. This contradiction shows that $[\omega^n] \neq 0$. Next, if $[\omega^k]$ were zero for some $k \in \{1, \dots, n-1\}$, then

$$[\omega^n] = [\omega^k] \wedge [\omega^{n-k}] = 0,$$

which contradicts what we just proved. $\square$

Example 1.2.9. For all $n \geq 1$, the sphere S^{2n} admits a symplectic form if and only if $n = 1$. Clearly, since S^2 is an orientable surface, it is symplectic by Example 1.2.4. Noting that

$$H^k(S^{2n}) = \begin{cases} \mathbb{R} & \text{if } k = 0 \text{ or } k = 2n \\ 0 & \text{otherwise} \end{cases},$$

we see that $H^2(S^{2n}) = 0$ when $n \neq 1$. By Proposition 1.2.8, S^{2n} is not symplectic for $n > 1$.

1.3 Darboux-Moser theorems

In this section, we discuss some local theory of symplectic manifolds. An important result is Moser's trick, which is a useful tool in deciding whether symplectic structures are equivalent. It will allow us to prove the Darboux theorem, which establishes a local normal form for symplectic manifolds.

Definition 1.3.1. Let M be a manifold. An isotopy is a smooth family $\{\rho_t\}_{t \in I}$ of diffeomorphisms of M, where I is an invertal containing 0 and $\rho_0 = \mathrm{Id}_M$. Stated otherwise, an isotopy is a smooth map $\rho : I \times M \to M$, such that for each $t \in I$, the map $\rho_t : M \to M : x \mapsto \rho(t, x)$ is a diffeomorphism and $\rho_0 = \mathrm{Id}_M$.

Definition 1.3.2. A time-dependent vector field on M is a family $\{X_t\}_{t \in I}$ of vector fields on M, depending smoothly on t.

Remark 1.3.3. An isotopy $\{\rho_t\}_{t \in I}$ determines a unique time-dependent vector field $\{X_t\}_{t \in I}$ defined by

$$X_t(p) = \left.\frac{d}{ds}\right|_{s=t} \rho_s(q),$$

where $q = \rho_t^{-1}(p)$. That is, X_t satisfies

$$X_t \circ \rho_t = \frac{d}{dt}\rho_t. \tag{1.2}$$

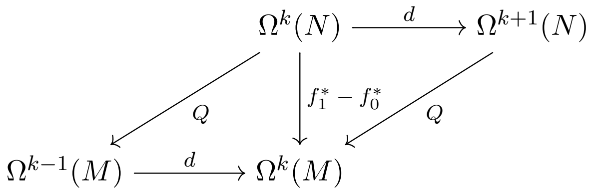
Conversely, given a time-dependent vector field $\{X_t\}_{t \in I}$, there exists a local isotopy ρ that solves the ODE (1.2) with initial condition $\rho_0 = \mathrm{Id}$. Note that in general, ρ is only defined on an open subset of $I \times M$. However, if M is compact, or more generally if the X_t are compactly supported, then the solution ρ is globally defined on $I \times M$.

We briefly recall some more useful facts.

Definition 1.3.4. Let $f_0, f_1 : M \to N$ be smooth maps between smooth manifolds. A homotopy operator between f_0 and f_1 is a linear map $Q : \Omega^\bullet(N) \to \Omega^{\bullet-1}(M)$ such that

$$f_1^* - f_0^* = d \circ Q + Q \circ d$$

in the diagram



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Let $\{\rho_t\}_{t\in[0,1]}$ be an isotopy on M with corresponding time-dependent vector field $\{X_t\}_{t\in[0,1]}$. If we define

$$Q : \Omega^k(M) \to \Omega^{k-1}(M) : \alpha \mapsto \int_0^1 \rho_t^*(\iota_{X_t}\alpha)dt, \tag{1.3}$$

then we have the homotopy formula

$$\rho_1^*\alpha - \alpha = dQ(\alpha) + Q(d\alpha). \tag{1.4}$$

Indeed, we compute

$$\begin{aligned} Q(d\alpha) + dQ(\alpha) &= \int_0^1 \rho_t^*\left(\iota_{X_t} d\alpha\right) dt + d\int_0^1 \rho_t^*\left(\iota_{X_t}\alpha\right) dt \\ &= \int_0^1 \rho_t^*\left(\iota_{X_t} d\alpha + d\iota_{X_t}\alpha\right) dt \\ &= \int_0^1 \rho_t^* \pounds_{X_t}\alpha \\ &= \int_0^1 \frac{d}{dt}\rho_t^*\alpha \\ &= \rho_1^*\alpha - \rho_0^*\alpha \\ &= \rho_1^*\alpha - \alpha. \end{aligned}$$

In the above manipulations, we used Cartan's magic formula (Lemma 8.2.1 in the appendix) and Lemma 8.2.2 in the appendix.

Definition 1.3.5. A smooth homotopy between maps $f_0, f_1 : M \to N$ is a smooth map $h : [0,1] \times M \to N$ such that $h(0,\cdot) = f_0$ and $h(1,\cdot) = f_1$.

Proposition 1.3.6. *Let $f_0, f_1 : M \to N$ be homotopic maps. Then there exists a homotopy operator $\tilde{Q}$ between them.*

Proof. Let $h : [0,1]\times M \to N$ be a smooth homotopy between f_0 and f_1. Consider the manifold $W = \mathbb{R}\times M$, and let t be the coordinate on $\mathbb{R}$. The vector field $\frac{\partial}{\partial t}$ on W is complete and its flow ϕ_s is given by $\phi_s(t,p) = (s+t,p)$. By the homotopy formula (1.4), we find $Q : \Omega^k(W) \to \Omega^{k-1}(W)$ such that $\phi_1^* - \phi_0^* = d\circ Q + Q\circ d$. On the other hand, denoting $i : M \hookrightarrow \mathbb{R}\times M : p \mapsto (0,p)$, we have

$$\begin{cases} f_0 = h(0,\cdot) = h\circ i \\ f_1 = h(1,\cdot) = h\circ\phi_1\circ i \end{cases}.$$

Hence,

$$\begin{aligned} f_1^* - f_0^* = i^*\circ\phi_1^*\circ h^* - i^*\circ h^* = i^*\circ(\phi_1^* - \phi_0^*)\circ h^* = i^*\circ(d\circ Q + Q\circ d)\circ h^* \\ = d\circ(i^*\circ Q\circ h^*) + (i^*\circ Q\circ h^*)\circ d. \end{aligned}$$

The proof ends by defining $\tilde{Q} := i^*\circ Q\circ h^*$. □

Remark 1.3.7. It follows that for homotopic maps f_0 and f_1, the induced maps on cohomology

$$[f_i^*] : H^k(N) \to H^k(M) : [\alpha] \mapsto [f_i^*\alpha]$$

for $i = 0, 1$ are equal. Indeed, for a closed form $\alpha \in \Omega^k(N)$, we have

$$f_1^*\alpha - f_0^*\alpha = d(\tilde{Q}(\alpha)) + \tilde{Q}(d(\alpha)) = d(\tilde{Q}(\alpha)),$$

hence $[f_0^*\alpha] = [f_1^*\alpha]$.

We now state the Tubular Neighborhood Theorem, which is a useful tool when working locally near a submanifold. It reduces analysis near the submanifold to analysis in a vector bundle, which is often preferable as one can use linear algebra in the fibers.

Let M be a manifold, $X \subset M$ a submanifold and $i : X \hookrightarrow M$ the inclusion map. Via the linear inclusions $d_x i : T_x X \to T_x M$, we consider $T_x X$ as a subspace of $T_x M$ for each $x \in X$. The quotient spaces $N_x X := T_x M / T_x X$ are the fibers of the normal bundle

$$NX := \frac{TM|_X}{TX} = \{(x, v) : \ x \in X, \ v \in N_x X\}.$$

Denote the zero section of NX by $i_0 : X \hookrightarrow NX$. A neighborhood U_0 of the zero section X in NX is called convex if the intersection $U_0 \cap N_x X$ with each fiber is convex.

Theorem 1.3.8 (Tubular Neighborhood Theorem)**.** *In the above setup, there exists a convex neighborhood U_0 of X in NX, a neighborhood U of X in M (called tubular neighborhood), and a diffeomorphism $\phi : U_0 \to U$ such that the following diagram commutes:*

$$\begin{array}{ccc} NX \supseteq U_0 & \xrightarrow[\cong]{\phi} & U \subseteq M \\ & {\scriptstyle i_0}\nwarrow & \uparrow {\scriptstyle i} \\ & & X \end{array}.$$

Proof. See [Ca]. □

The Tubular Neighborhood Theorem is a key ingredient of the Relative Poincaré Lemma.

Proposition 1.3.9 (Relative Poincaré Lemma)**.** *Let $X \subset M$ be a submanifold and denote by $i : X \hookrightarrow M$ the inclusion. Let U be a tubular neighborhood of X. If $\beta \in \Omega^k(U)$ is closed and $i^*\beta = 0$, then there exists $\eta \in \Omega^{k-1}(U)$ such that*

$$\begin{cases} d\eta = \beta \\ \eta_x = 0 \quad \text{for all } x \in X \end{cases}.$$

Proof. Via the diffeomorphism $\phi : U_0 \subset NX \to U \subset M$ from the Tubular Neighborhood Theorem, it is equivalent to work in U_0. Let $j : X \hookrightarrow U_0$ denote the zero section and $\pi : U_0 \to X$ the bundle projection. We define a retraction r of U_0 onto X by

$$r : [0,1] \times U_0 \to U_0 : (t, x, v) \mapsto (x, tv),$$

which is well-defined by convexity of U_0. Note that $r_1 = \mathrm{Id}_{U_0}$ and $r_0 = j \circ \pi$ are homotopic through r. Hence by Proposition 1.3.6, we find a homotopy operator $\tilde{Q}$, which gives

$$\beta - \pi^*(j^*\beta) = d\tilde{Q}(\beta) + \tilde{Q}(d\beta).$$

Since β is closed and $j^*\beta = 0$, we get that $\beta = d\tilde{Q}(\beta)$. This makes us set $\eta := \tilde{Q}(\beta)$. It remains to check that η vanishes on X. From the construction of $\tilde{Q}$ in Proposition 1.3.6 and definition (1.3), we see that it is enough to show that

$$\iota_{\frac{\partial}{\partial t}}(r^*\beta)$$

is zero on the slices $\{s\}\times X$ for $s \in [0,1]$. We work in a local trivialization of U_0 with coordinates $(x_1,\ldots,x_n,v_1,\ldots,v_m)$, where m is the codimension of X in M and $v_1,\ldots,v_m$ are coordinates in the fibers of U_0. In these coordinates, r is given by

$$r : (t, x_1,\ldots,x_n,v_1,\ldots,v_m) \mapsto (x_1,\ldots,x_n,tv_1,\ldots,tv_m).$$

This implies that

$$r_*\left(\frac{\partial}{\partial t}\right) = \sum_{i=1}^{n} \frac{\partial x_i}{\partial t}\frac{\partial}{\partial x_i} + \sum_{i=1}^{m} \frac{\partial (tv_i)}{\partial t}\frac{\partial}{\partial v_i} = \sum_{i=1}^{m} v_i \frac{\partial}{\partial v_i},$$

which vanishes on $X \leftrightarrow \{v_1 = \cdots = v_m = 0\}$. It follows that $\iota_{\frac{\partial}{\partial t}}(r^*\beta)$ vanishes on each slice $\{s\}\times X$, which finishes the proof. □

We now address the Moser stability theorem, which is a key result in the deformation theory of symplectic forms. It states that one cannot get new symplectic structures by deforming a given structure within its cohomology class.

Theorem 1.3.10 (Moser)**.** *Let M be a compact manifold and $\{\omega_t\}_{t\in[0,1]}$ a smooth family of symplectic forms such that $[\omega_t] \in H^2(M)$ is independent of t. Then there exists an isotopy $\rho : [0,1]\times M \to M$ such that $\rho_t^*\omega_t = \omega_0$ for all $t \in [0,1]$.*

Proof. Let $\rho : [0,1]\times M \to M$ be an isotopy with associated time dependent vector field $\{X_t\}_{t\in[0,1]}$. We have the following equivalences:

$$\begin{aligned}\rho_t^*\omega_t = \omega_0 \ \forall t \in [0,1] &\Leftrightarrow \frac{d}{dt}(\rho_t^*\omega_t) = 0\\ &\Leftrightarrow \rho_t^*\left(\pounds_{X_t}\omega_t + \frac{d}{dt}\omega_t\right) = 0\\ &\Leftrightarrow \pounds_{X_t}\omega_t + \frac{d}{dt}\omega_t = 0\\ &\Leftrightarrow d\left(\iota_{X_t}\omega_t\right) + \frac{d}{dt}\omega_t = 0.\end{aligned}$$

In the above manipulations, we used Lemma 8.2.3 in the appendix, injectivity of the linear maps ρ_t^* and Cartan's magic formula (Lemma 8.2.1 in the appendix) along with closedness of the ω_t. Hence, the theorem asks us to find $\{X_t\}_{t\in[0,1]}$ such that $d\left(\iota_{X_t}\omega_t\right) + \frac{d}{dt}\omega_t = 0$.
Note that the map

$$\pi : \Omega^2(M)_{\text{closed}} \to H^2(M) : \omega \mapsto [\omega]$$

is linear, which implies that

$$\frac{d}{dt}[\omega_t] = \frac{d}{dt}\left(\pi(\omega_t)\right) = \pi_*\left(\frac{d}{dt}\omega_t\right) = \pi\left(\frac{d}{dt}\omega_t\right) = \left[\frac{d}{dt}\omega_t\right].$$

Hence, the assumption $\left[\frac{d}{dt}\omega_t\right] = \frac{d}{dt}[\omega_t] = 0$ yields $\mu_t \in \Omega^1(M)$ for $t \in [0,1]$ such that $d\mu_t = \frac{d}{dt}\omega_t$. With some extra work, one shows that the one-forms μ_t can be chosen in a smooth way [MS, p.95]. Hence,

$$d\left(\iota_{X_t}\omega_t\right) + \frac{d}{dt}\omega_t = 0 \Leftrightarrow d\left(\iota_{X_t}\omega_t + \mu_t\right)) = 0.$$

Consequently, it is enough to choose X_t such that

$$\iota_{X_t}\omega_t + \mu_t = 0. \tag{1.5}$$

Non-degeneracy of ω_t implies that the map

$$\omega_t^\flat : \mathfrak{X}(M) \to \Omega^1(M) : X \mapsto \iota_X \omega_t$$

is an isomorphism. So we can set $X_t := -(\omega_t^\flat)^{-1}(\mu_t)$ for $t \in [0,1]$. By compactness of M, we can integrate X_t to an isotopy $\rho : [0,1] \times M \to M$, which by construction satisfies the statement of the theorem. □

Remark 1.3.11. The Moser theorem along with Remark 1.3.7 settles the deformation theory of symplectic manifolds. Indeed, let (M, ω) be a compact symplectic manifold and let $\alpha \in \Omega^2(M)$ with $d\alpha = 0$. For small enough t, we have that $\omega_t := \omega + t\alpha$ is a curve of symplectic forms with tangent α at $t = 0$. The aforementioned results ensure that the existence of an isotopy $\{\rho_t\}$ satisfying $\rho_t^* \omega_t = \omega$ is equivalent with α being exact. This implies that $H^2(M)$ is the "tangent space" to the moduli space of deformations of the symplectic structure. Heuristically, deformations of symplectic forms are classified by the second de Rham cohomology group.

We will now prove a local version of the Moser Theorem 1.3.10. When working with isotopies, the tube lemma from topology is often useful.

Lemma 1.3.12 (Tube Lemma)**.** *Let X and Y be topological spaces and assume that Y is compact. If N is an open subset of $X \times Y$ containing the slice $\{x_0\} \times Y$, then N contains some tube $W \times Y$ about $\{x_0\} \times Y$, where W is an open neighborhood of x_0 in X.*

Proof. See [Mun]. □

We will use the Tube Lemma in the following form:

Lemma 1.3.13. *Let M be a topological space and let $\{U_t\}_{t\in[0,1]}$ be a family of subsets of M, such that $\bigcup_{t\in[0,1]} (\{t\} \times U_t)$ is open in $[0,1] \times M$. Then $\bigcap_{t\in[0,1]} U_t$ is open in M.*

Proof. If $\bigcap_{t\in[0,1]} U_t = \emptyset$, there is nothing to prove. Let $m \in \bigcap_{t\in[0,1]} U_t$. We have that $\bigcup_{t\in[0,1]} (\{t\} \times U_t)$ is open in $[0,1] \times M$, containing the slice $[0,1] \times \{m\}$. By the Tube Lemma, we find an open V in M around m such that $[0,1] \times V$ is contained inside $\bigcup_{t\in[0,1]} (\{t\} \times U_t)$. This implies that V is an open neighborhood of m, contained in $\bigcap_{t\in[0,1]} U_t$, which proves that $\bigcap_{t\in[0,1]} U_t$ is open. □

Theorem 1.3.14 (Local Moser)**.** *Let M be a manifold and $X \subset M$ a submanifold. Let ω_0 and ω_1 be symplectic forms on M such that $\omega_0|_p = \omega_1|_p$ for all $p \in X$. Then there exist tubular neighborhoods U_0, U_1 of X and a diffeomorphism $f : U_0 \to U_1$ such that $f|_X = Id_X$ and $f^* \omega_1 = \omega_0$.*

Proof. Choose a tubular neighborhood U_0 of X. The 2-form $\omega_1 - \omega_0$ on U_0 is closed, and $(\omega_1 - \omega_0)_p = 0$ for all $p \in X$. By the relative Poincaré lemma (Proposition 1.3.9), there exists $\eta \in \Omega^1(U_0)$ such that $\omega_1 - \omega_0 = d\eta$ and $\eta_p = 0$ at all $p \in X$. Now consider for $0 \leq t \leq 1$ the straight line homotopy

$$\omega_t := \omega_0 + t(\omega_1 - \omega_0) = \omega_0 + td\eta,$$

consisting of closed 2-forms ω_t on U_0. Note that $\omega_t|_p = \omega_0|_p$ is non-degenerate for all $p \in X$. Since non-degeneracy is an open property, there exists an open neighborhood U of X on which ω_t is non-degenerate for all $t \in [0,1]$ (use the Tube Lemma). Shrinking U_0 if necessary, we may assume that $\{\omega_t\}_{t\in[0,1]}$ is a smooth family of symplectic forms on U_0. As in Theorem 1.3.10, it now suffices to solve the Moser equation (1.5). Noting that $\frac{d}{dt}\omega_t = \omega_1 - \omega_0 = d\eta$, we have to solve the equation

$$\iota_{v_t} \omega_t = -\eta$$

for v_t. That is, using non-degeneracy of ω_t, we define $v_t := -(\omega_t^\flat)^{-1}(\eta)$. Since $\eta_p = 0$ for all $p \in X$, also v_t vanishes on X. We now argue that the isotopy ρ that integrates v_t is defined on $[0,1] \times V$, where V is some open neighborhood of X. Define for each $t \in [0,1]$ the set $V_t := \{p \in M : \rho_t(p) \text{ is defined}\}$. Note that $X \subset V_t$ for each t, since v_t vanishes on X. Also, $\bigcup_{t\in[0,1]} (\{t\} \times V_t)$ (which is the domain of ρ) is open in $[0,1] \times M$. By Lemma 1.3.13, $V := \bigcap_{t\in[0,1]} V_t$ is an open neighborhood of X, and ρ is defined on $[0,1] \times V$. Again shrinking U_0 if necessary, we assume that $\rho : [0,1] \times U_0 \to M$ with $\rho_t^* \omega_t = \omega_0$ for all $t \in [0,1]$. Moreover, since $v_t|_X = 0$, it follows that $\rho_t|_X = \mathrm{Id}_X$. The proof ends by defining $f := \rho_1$ and $U_1 := \rho_1(U_0)$. □

We can now prove the Darboux Theorem, which states that symplectic manifolds (of equal dimension) all look the same locally.

Theorem 1.3.15 (Darboux)**.** *Let (M^{2n}, ω) be a symplectic manifold and let $x \in M$. Then there exists a coordinate system $(U, q_1, \dots, q_n, p_1, \dots, p_n)$ centered at x such that on U:*

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

Proof. The standard form for skew-symmetric bilinear maps (Proposition 8.1.1 in appendix) gives a basis $\{v_1, w_1, \dots, v_n, w_n\}$ of the symplectic vector space T_xM so that $\omega_x \in \wedge^2 T_x^*M$ has the canonical form. If $(q_1', p_1', \dots, q_n', p_n')$ are the corresponding linear coordinates on T_xM, then we have

$$\omega_x = \sum_{i=1}^{n} dq_i' \wedge dp_i'.$$

Fix a Riemannian metric on M and denote by ϕ the exponential map $\phi := \exp_x$. This is a local diffeomorphism between an open $V \subset T_xM$ around the origin 0_x and an open $U \subset M$ around x. Moreover, $\phi(0_x) = \exp_x(0_x) = x$ and $(d\phi)_{0_x} = (d\exp_x)_{0_x} = \mathrm{Id}_{T_xM}$. On V, we consider the symplectic forms $\omega_0 := \omega_x$ and $\omega_1 := \phi^*\omega$. Note that for $v, w \in T_{0_x}V \cong T_xM$:

$$\omega_1|_{0_x}(v,w) = \omega|_{\phi(0_x)}\big((d\phi)_{0_x}(v), (d\phi)_{0_x}(w)\big) = \omega_x(v,w) = \omega_0|_{0_x}(v,w),$$

hence $\omega_0|_{0_x} = \omega_1|_{0_x}$. We now apply the local Moser theorem to the submanifold $\{0_x\} \subset V$: this gives open neighborhoods U_0 and U_1 of 0_x and a diffeomorphism $f : U_0 \to U_1$ that satisfies $f(0_x) = 0_x$ and $f^*\omega_1 = \omega_0$. Hence, $(\phi \circ f)^*\omega = \omega_0$, which implies that on the open subset $\phi(U_1)$ around x:

$$\begin{aligned}\omega = (f^{-1} \circ \phi^{-1})^*\omega_0 &= \big(f^{-1} \circ \phi^{-1}\big)^* \left(\sum_{i=1}^{n} dq_i' \wedge dp_i'\right) \\ &= \sum_{i=1}^{n} d\left(q_i' \circ f^{-1} \circ \phi^{-1}\right) \wedge d\left(p_i' \circ f^{-1} \circ \phi^{-1}\right).\end{aligned}$$

Setting new coordinates $q_i := q_i' \circ f^{-1} \circ \phi^{-1}$ and $p_i := p_i' \circ f^{-1} \circ \phi^{-1}$ completes the proof. □

We will need a generalization of Darboux' theorem to the case of closed two-forms with constant rank. The proof below uses some concepts that are introduced in the last section of Chapter 2.

Theorem 1.3.16 (Darboux)**.** *[AM, Theorem 5.1.3] Let M be a $(2n+k)$-dimensional manifold and ω a closed 2-form of constant rank $2n$. For each $x_0 \in M$, there is a coordinate chart $(U, x_1, \dots, x_n, y_1, \dots, y_n, w_1, \dots, w_k)$ about x_0 such that*

$$\omega|_U = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Proof. Let us first show that $\mathrm{Ker}(\omega)$ is a completely integrable distribution. Consider the bundle map $\omega^\flat : TM \to T^*M : v \mapsto \iota_v\omega$. By assumption, this map has constant rank $2n$, which implies that its kernel $\mathrm{Ker}(\omega)$ is a smooth rank k subbundle of TM ([Lee, Theorem 10.34]). That is, $\mathrm{Ker}(\omega)$ is a smooth regular distribution. To show it is completely integrable, it is enough to check involutivity by Frobenius' theorem. If $X, Y \in \Gamma\big(\mathrm{Ker}(\omega)\big)$ and $Z \in \Gamma(TM)$, then

$$\begin{aligned} 0 = d\omega(X,Y,Z) &= X\big(\omega(Y,Z)\big) - Y\big(\omega(X,Z)\big) + Z\big(\omega(X,Y)\big) \\ &\quad - \omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X) \\ &= -\omega([X,Y],Z). \end{aligned}$$

Hence $[X,Y] \in \Gamma\big(\mathrm{Ker}(\omega)\big)$, and the distribution $\mathrm{Ker}(\omega)$ is completely integrable.
Now let $x_0 \in M$. Since $\mathrm{Ker}(\omega)$ is a completely integrable k-dimensional distribution, we can find coordinates $(U, p_1, \ldots, p_n, q_1, \ldots, q_n, w_1, \ldots, w_k)$ centered at x_0 so that $\{\frac{\partial}{\partial w_1}, \ldots, \frac{\partial}{\partial w_k}\}$ is a local basis for $\mathrm{Ker}(\omega)$ on U. Denote by N the slice given by $w_1 = \cdots = w_k = 0$, which is a $2n$-dimensional submanifold with coordinates $(U \cap N, p_1, \ldots, p_n, q_1, \ldots, q_n)$ centered at x_0. If $i : N \to M$ denotes the inclusion, then $i^*\omega$ is a closed 2-form on N of maximal rank $2n$: it is a symplectic form on N. By the Darboux Theorem 1.3.15, shrinking U if necessary, we find new coordinates $(U \cap N, x_1, \ldots, x_n, y_1, \ldots, y_n)$ on N near x_0 so that

$$i^*\omega = \sum_{i=1}^n dx_i \wedge dy_i.$$

Extending the x_i and y_j locally near x_0, we get that $(x_1, \ldots, x_n, y_1, \ldots, y_n, w_1, \ldots, w_k)$ is a coordinate system for M around x_0, since the Jacobian determinant of the map

$$(p_1, \ldots, p_n, q_1, \ldots, q_n, w_1, \ldots, w_k) \mapsto (x_1, \ldots, x_n, y_1, \ldots, y_n, w_1, \ldots, w_k)$$

is

$$\begin{vmatrix} \left[\frac{\partial(x_i,y_j)}{\partial(p_i,q_j)}\right] & \star \\ 0 & I \end{vmatrix} = \left| \left[\frac{\partial(x_i,y_j)}{\partial(p_i,q_j)}\right] \right|,$$

which is non-vanishing at x_0. In these coordinates, the expression for ω does not involve $dw_1, \ldots, dw_k$, whence

$$\omega = \sum_{i=1}^n dx_i \wedge dy_i.$$

$\square$

Remark 1.3.17. A manifold M endowed with a closed two-form ω of constant rank is called a presymplectic manifold. Theorem 1.3.15 is the Darboux theorem for presymplectic manifolds.

Chapter 2

Preliminary Poisson Geometry

An extension of symplectic geometry is provided by Poisson geometry. Poisson geometry indeed started off as an outgrowth of symplectic geometry, though nowadays it is an extensive theory that bears connection with many other branches in mathematics. Poisson manifolds arise naturally as phase spaces of classical particles, but they are also entangled with non-commutative geometry and integrable systems, to name a few.

This thesis addresses log-symplectic manifolds, which form a convenient class of Poisson manifolds that can be considered as an intermediate level between the symplectic world and the generic Poisson world.

In this preliminary chapter, the main features of Poisson geometry are presented, with emphasis on the aspects that will be of particular interest for us. Since the author was not familiar with Poisson geometry prior to writing this thesis, he chose to make this chapter into a rather detailed introduction, as a personal exercise. Readers already familiar with Poisson geometry can of course ignore this chapter, or at least the details of it. What follows is mainly a compilation of results from [FM], [DT],[CW] and [LPV].

2.1 Almost Poisson structures (1)

Definition 2.1.1. An almost Poisson structure on a smooth manifold M is a bilinear bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ that satisfies

(i) Skew-symmetry: $\{f, g\} = -\{g, f\}$;

(ii) Leibniz identity: $\{f, gh\} = \{f, g\}h + g\{f, h\}$.

The Leibniz identity says that $\{f, \cdot\}$ is a derivation of $C^\infty(M)$. By skew-symmetry, the bracket $\{\cdot,\cdot\}$ is a derivation in both arguments.

Example 2.1.2. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a finite dimensional Lie algebra. Its dual $\mathfrak{g}^*$ inherits a canonical almost Poisson bracket. Given $f \in C^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$, the differential

$$d_\xi f : T_\xi\mathfrak{g}^* \cong \mathfrak{g}^* \to T_{f(\xi)}\mathbb{R} \cong \mathbb{R}$$

can be viewed as a map $\mathfrak{g}^* \to \mathbb{R}$. So it is an element of $\mathfrak{g}^{**}$. Since $\mathfrak{g}$ is finite dimensional, we have $\mathfrak{g}^{**} \cong \mathfrak{g}$, so we can consider $d_\xi f \in \mathfrak{g}$. This allows us to define an almost Poisson bracket $\{\cdot,\cdot\}$ on $C^\infty(\mathfrak{g}^*)$ as follows: for $f, h \in C^\infty(\mathfrak{g}^*)$ we define $\{f, h\} \in C^\infty(\mathfrak{g}^*)$ by

$$\{f, h\}(\xi) = \langle [d_\xi f, d_\xi h], \xi\rangle \quad \text{for } \xi \in \mathfrak{g}^*,$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. One checks that this bracket has the desired properties.

Almost Poisson structures are local. In order to show this, we need a lemma.

Lemma 2.1.3. *Let* $\{\cdot,\cdot\}$ *be an almost Poisson structure on* M*. Then for all* $f, g \in C^\infty(M)$*:*

$$supp\{f,g\} \subset supp(f) \cap supp(g).$$

Proof. We show that $\operatorname{supp}\{f,g\} \subset \operatorname{supp}(f)$. If $\operatorname{supp}(f) = M$, then there is nothing to prove. So assume that $\operatorname{supp}(f) \neq M$. Choose $x_0 \notin \operatorname{supp}(f)$. The open sets $V := M \setminus \{x_0\}$ and $U := M \setminus \operatorname{supp}(f)$ cover M. Choose a partition of unity ρ_U, ρ_V subordinate to the cover $\{U, V\}$. Then we have:

$$\begin{aligned}
\{f,g\}(x_0) &= \{\rho_U f + \rho_V f, g\}(x_0) && (\text{since } \rho_U + \rho_V \equiv 1)\\
&= \{\rho_V f, g\}(x_0) && (\rho_U f \equiv 0 \text{ since } \operatorname{supp}(\rho_U) \subset M \setminus \operatorname{supp}(f))\\
&= \rho_V(x_0)\{f,g\}(x_0) + f(x_0)\{\rho_V, g\}(x_0) && (\text{Leibniz identity})\\
&= 0. && (f(x_0) = \rho_V(x_0) = 0)
\end{aligned}$$

This shows that $M \setminus \operatorname{supp}(f) \subset \{x \in M : \{f,g\}(x) = 0\}$. Taking complements in this inclusion, we get $\{x \in M : \{f,g\}(x) \neq 0\} \subset \operatorname{supp}(f)$. Taking closures then gives $\operatorname{supp}\{f,g\} \subset \operatorname{supp}(f)$. □

Corollary 2.1.4. *Given an almost Poisson structure* $\{\cdot,\cdot\}$ *on* M*, we can restrict the bracket to an open subset* $U \subset M$*, obtaining an almost Poisson bracket* $\{\cdot,\cdot\}_U$ *such that for* $f, g \in C^\infty(M)$*, we have*

$$\{f,g\}|_U = \{f|_U, g|_U\}_U.$$

Proof. We show that the formula $\{f,g\}|_U = \{f|_U, g|_U\}_U$ yields a well-defined bracket on U. Take $\alpha, \beta \in C^\infty(U)$ such that $\alpha = f|_U = f'|_U$ and $\beta = g|_U = g'|_U$ for $f, f', g, g' \in C^\infty(M)$. Note that $(g - g')|_U \equiv 0$, hence $\{x \in M : (g-g')(x) \neq 0\} \subset M \setminus U$. Then $\operatorname{supp}(g - g') \subset M \setminus U$ by taking closures. Now Lemma 2.1.3 implies that

$$\operatorname{supp}\{f, g' - g\} \subset \operatorname{supp}(f) \cap \operatorname{supp}(g - g') \subset M \setminus U.$$

Hence $\{f, g' - g\}|_U \equiv 0$, which implies that $\{f, g'\}|_U = \{f,g\}|_U$. Similarly, $\{f', g\}|_U = \{f,g\}|_U$. □

Remark 2.1.5. Corollary 2.1.4 shows in particular that the value of $\{f,g\}$ at some point $x \in M$ only depends on the restriction of f and g to a neighborhood of x. Henceforth, we will no longer distinguish between $\{\cdot,\cdot\}$ and $\{\cdot,\cdot\}_U$.

The description of almost Poisson structures in terms of a bracket on the algebra of smooth functions is not always the most efficient one. There is an alternative description in terms of so-called bivector fields. A brief excursion to multivector fields is needed.

2.2 Multivector fields

Let M be a smooth n-dimensional manifold and k a positive integer. Recall that the smooth differential k-forms $\Omega^k(M)$ are sections of the vector bundle $\wedge^k T^*M$. They can be identified with the $C^\infty(M)$-multilinear, alternating maps

$$\omega : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M).$$

The k-multivector fields, denoted by $\mathfrak{X}^k(M)$, are sections of the dual bundle $(\wedge^k T^*M)^* = \wedge^k (T^*M)^* = \wedge^k TM$. They can be identified with the $C^\infty(M)$-multilinear, alternating maps

$$\nu : \Omega^1(M) \times \cdots \times \Omega^1(M) \to C^\infty(M).$$

Let $(x_1, \dots, x_n)$ be a system of local coordinates on M. Then $\omega \in \Omega^k(M)$ and $\nu \in \mathfrak{X}^k(M)$ have local expressions

$$\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1,\dots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \quad \text{and} \quad \nu = \sum_{1 \le i_1 < \cdots < i_k \le n} \nu_{i_1,\dots,i_k} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}}.$$

The pairing $\langle \omega, \nu \rangle$ of ω and ν is the function defined by

$$\langle \omega, \nu \rangle = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1,\dots,i_k} \nu_{i_1,\dots,i_k}.$$

One checks that this definition does not depend on the choice of coordinates.

The space of all multivector fields $\mathfrak{X}^\bullet(M) := \bigoplus_{k=0}^n \mathfrak{X}^k(M)$ is endowed with the usual operations, listed below.

Wedge product

For $p \in M$, the exterior algebra $\wedge T_pM$ of the vector space T_pM has a wedge product $\wedge$. It is defined by

$$\begin{aligned} &\wedge : \wedge^k T_pM \times \wedge^l T_pM \to \wedge^{k+l} T_pM : \\ &\quad (v \wedge w)(\alpha_1, \dots, \alpha_{k+l}) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} sgn(\sigma) v(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(k)}) w(\alpha_{\sigma(k+1)}, \dots, \alpha_{\sigma(k+1)}), \end{aligned}$$

where $\alpha_1, \dots, \alpha_{k+l} \in T_p^*M$. It induces a wedge product of multivector fields by

$$\wedge : \mathfrak{X}^k(M) \times \mathfrak{X}^s(M) \to \mathfrak{X}^{k+s}(M) : \ (\nu \wedge \zeta)_p = \nu_p \wedge \zeta_p, \quad \text{where } \nu_p, \zeta_p \in \wedge T_pM.$$

With the convention that $\mathfrak{X}^0(M) = C^\infty(M)$ and $f \wedge \nu = f\nu$ for $f \in C^\infty(M), \nu \in \mathfrak{X}^k(M)$, the wedge product turns $\mathfrak{X}^\bullet(M)$ into a Grassmann algebra. That is,

(i) $(f\nu_1 + g\nu_2) \wedge \zeta = f\nu_1 \wedge \zeta + g\nu_2 \wedge \zeta \quad$ for $f, g \in C^\infty(M)$;

(ii) $\nu \wedge \zeta = (-1)^{kl} \zeta \wedge \nu \quad$ for $\nu \in \mathfrak{X}^k(M)$ and $\zeta \in \mathfrak{X}^l(M)$;

(iii) $(\nu_1 \wedge \nu_2) \wedge \nu_3 = \nu_1 \wedge (\nu_2 \wedge \nu_3)$.

We can evaluate wedge products by

$$(X_1 \wedge \cdots \wedge X_k)(\alpha_1, \dots, \alpha_k) = \det[\alpha_i(X_j)]_{i,j}$$

for $X_1, \dots, X_k \in \mathfrak{X}(M)$ and $\alpha_1, \dots, \alpha_k \in \Omega^1(M)$.

Interior product

Given a k-vectorfield $\nu \in \mathfrak{X}^k(M)$ and a 1-form $\alpha \in \Omega^1(M)$, the interior product of ν by α is $\iota_\alpha \nu \in \mathfrak{X}^{k-1}(M)$, defined by

$$\iota_\alpha \nu(\alpha_1, \dots, \alpha_{k-1}) = \nu(\alpha, \alpha_1, \dots, \alpha_{k-1}).$$

It is a degree -1 derivation of $\wedge$, satisfying:

(i) $\iota_\alpha(f\nu_1 + g\nu_2) = f\iota_\alpha\nu_1 + g\iota_\alpha\nu_2$;

(ii) $\iota_\alpha(\nu \wedge \xi) = (\iota_\alpha \nu) \wedge \xi + (-1)^k \nu \wedge (\iota_\alpha \xi)$ for $\nu \in \mathfrak{X}^k(M)$;

(iii) $\iota_{(f\alpha + g\beta)}\nu = f\iota_\alpha\nu + g\iota_\beta\nu$.

Pushforward

If $\Phi : M \to N$ is a smooth map, we have an induced pullback map $\Phi^* : \Omega^\bullet(N) \to \Omega^\bullet(M)$ on differential forms. The pushforward is the dual operation on multivector fields. One has to be more careful though; as opposed to the pullback, the pushforward is not always defined.

For each $p \in M$ we have a linear map $d_p\Phi : T_pM \to T_{\Phi(p)}N$ and an induced linear map

$$d_p\Phi : \wedge^k T_pM \to \wedge^k T_{\Phi(p)}N : v_1 \wedge \cdots \wedge v_k \mapsto (d_p\Phi)(v_1) \wedge \cdots \wedge (d_p\Phi)(v_k),$$

where $d_p\Phi$ is the identity map on $\mathbb{R} = \wedge^0 T_pM = \wedge^0 T_{\Phi(p)}N$. Two k-vectorfields $\nu \in \mathfrak{X}^k(M)$ and $\xi \in \mathfrak{X}^k(N)$ are said to be Φ-related if $\xi_{\Phi(p)} = (d_p\Phi)(\nu_p)$ for all $p \in M$. In general, this relation does not define a map $\mathfrak{X}^k(M) \to \mathfrak{X}^k(N)$. We can have several multivector fields on N that are Φ-related to a fixed multivector field on M. For instance, consider

$$\Phi : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x, 0).$$

Saying that $Y \in \mathfrak{X}(\mathbb{R}^2)$ is Φ-related to $X \in \mathfrak{X}(\mathbb{R}^2)$ only determines Y on $\{(x, 0) : x \in \mathbb{R}\}$. It is also possible that there exist no multivector fields that are Φ-related to a fixed multivector field. For instance, take $X \in \mathfrak{X}(\mathbb{R}^2)$ in the previous example such that $(d_{(x,1)}\Phi)(X_{(x,1)})$ and $(d_{(x,0)}\Phi)(X_{(x,0)})$ are different. If Φ is a diffeomorphism however, we get a well-defined pushforward map

$$\Phi_* : \mathfrak{X}^k(M) \to \mathfrak{X}^k(N) : \nu \mapsto \Phi_*\nu,$$

where $(\Phi_*\nu)_{\Phi(p)} = (d_p\Phi)(\nu_p)$ for all $p \in M$. The pushforward satisfies

(i) $\Phi_*(a\nu_1 + b\nu_2) = a\Phi_*(\nu_1) + b\Phi_*(\nu_2)$ for $a, b \in \mathbb{R}$;

(ii) $\Phi_*(\nu \wedge \zeta) = \Phi_*(\nu) \wedge \Phi_*(\zeta)$.

Remark 2.2.1. Multivector fields $\nu \in \mathfrak{X}^k(M)$ and $\xi \in \mathfrak{X}^k(N)$ being Φ-related means that

$$\xi_{\Phi(p)}(\alpha_1, \ldots, \alpha_k) = \nu_p(\Phi^*\alpha_1, \ldots, \Phi^*\alpha_k) \text{ for all } \alpha_1, \ldots, \alpha_k \in T^*_{\Phi(p)}N.$$

Indeed,

$$\begin{aligned}
(d_p\Phi)(X_1|_p \wedge \cdots \wedge X_k|_p)(\alpha_1, \ldots, \alpha_k) &= (d_p\Phi)(X_1|_p) \wedge \cdots \wedge (d_p\Phi)(X_k|_p)(\alpha_1, \ldots, \alpha_k) \\
&= \begin{vmatrix} \alpha_1\,(d_p\Phi(X_1|_p)) & \ldots & \alpha_1\,(d_p\Phi(X_k|_p)) \\ \vdots & & \vdots \\ \alpha_k\,(d_p\Phi(X_1|_p)) & \ldots & \alpha_k\,(d_p\Phi(X_k|_p)) \end{vmatrix} \\
&= \begin{vmatrix} (\Phi^*\alpha_1)(X_1|_p) & \ldots & (\Phi^*\alpha_1)(X_k|_p) \\ \vdots & & \vdots \\ (\Phi^*\alpha_k)(X_1|_p) & \ldots & (\Phi^*\alpha_k)(X_k|_p) \end{vmatrix} \\
&= (X_1|_p \wedge \cdots \wedge X_k|_p)\big(\Phi^*\alpha_1, \ldots, \Phi^*\alpha_k\big).
\end{aligned}$$

Lie derivative

The Lie derivative of a k-vectorfield $\nu \in \mathfrak{X}^k(M)$ along a vectorfield $X \in \mathfrak{X}(M)$ is the k-vectorfield $\mathcal{L}_X\nu \in \mathfrak{X}^k(M)$ defined by

$$\mathcal{L}_X\nu = \frac{d}{dt}(\phi_{-t})_*\nu\bigg|_{t=0},$$

where ϕ is the flow of X. Note that the pushforward in this formula is well-defined, since the flow maps ϕ_{-t} are diffeomorphisms.
The Lie derivative is a degree 0 derivation of $\wedge$. If $X \in \mathfrak{X}(M)$ and $\nu_1, \nu_2 \in \mathfrak{X}^\bullet(M)$ then

(i) $\pounds_X(a\nu_1 + b\nu_2) = a\pounds_X\nu_1 + b\pounds_X\nu_2$ for $a, b \in \mathbb{R}$;

(ii) $\pounds_X(\nu_1 \wedge \nu_2) = (\pounds_X\nu_1) \wedge \nu_2 + \nu_1 \wedge (\pounds_X\nu_2)$;

(iii) $\pounds_X(f) = X(f)$ for $f \in C^\infty(M) = \mathfrak{X}^0(M)$;

(iv) $\pounds_X Y = [X, Y]$ for $Y \in \mathfrak{X}(M)$.

Schouten bracket

The Schouten bracket is an operation on multivector fields that extends the Lie bracket of vector fields. The following theorem ensures its existence and uniqueness.

Theorem 2.2.2 (Schouten bracket)**.** *There is a unique bilinear map $[\cdot,\cdot]$ that turns $\mathfrak{X}^{\bullet-1}(M)$ into a $\mathbb{Z}$-graded super Lie algebra, with the following properties:*

(i) For fixed $\xi \in \mathfrak{X}^k(M)$, the bracket $[\xi,\cdot]$ is a graded derivation of degree $k-1$ with respect to the wedge product on $\mathfrak{X}^\bullet(M)$.

(ii) For $X \in \mathfrak{X}(M)$, the bracket $[X,\cdot]$ is the Lie derivative.

(iii) For $\xi \in \mathfrak{X}^k(M)$ and $\zeta \in \mathfrak{X}^l(M)$, the value of $[\xi,\zeta]$ at a point p depends only on the restriction of ξ and ζ to a neighborhood of p.

Proof. See for instance Theorem 1.1 in [Vai]. □

Remark 2.2.3. Super Lie algebra means that skew-symmetry and the Jacobi identity hold with signs. That is:

- $[\nu,\zeta] = -(-1)^{(k-1)(l-1)}[\zeta,\nu]$ for $\nu \in \mathfrak{X}^k(M)$ and $\zeta \in \mathfrak{X}^l(M)$;
- $(-1)^{(k-1)(m-1)}[[\nu,\zeta],\tau] + (-1)^{(l-1)(k-1)}[[\zeta,\tau],\nu] + (-1)^{(m-1)(l-1)}[[\tau,\nu],\zeta] = 0$ for $\nu \in \mathfrak{X}^k(M)$, $\zeta \in \mathfrak{X}^l(M)$ and $\tau \in \mathfrak{X}^m(M)$.

Property (i) in Theorem 2.2.2 means that

$$[\xi, \zeta \wedge \tau] = [\xi,\zeta] \wedge \tau + (-1)^{(k-1)l}\zeta \wedge [\xi,\tau],$$

where $\xi \in \mathfrak{X}^k(M)$, $\zeta \in \mathfrak{X}^l(M)$ and $\tau \in \mathfrak{X}^m(M)$.
Property (ii) in Theorem 2.2.2 says that the Schouten bracket extends the Lie bracket, as was desired. Indeed, for $X, Y \in \mathfrak{X}(M)$ we get $[X,Y]_{Schouten} = \pounds_X Y = [X,Y]_{Lie}$.

Remark 2.2.4. Note that the grading for the graded Lie algebra structure on multivector fields differs from the grading for the graded algebra structure. Indeed, the Lie algebra grading is the algebra grading shifted by -1. Consequently, we have

$$[\cdot,\cdot] : \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \to \mathfrak{X}^{k+l-1}(M).$$

Indeed, $\nu \in \mathfrak{X}^k(M)$ has degree $k-1$ and $\zeta \in \mathfrak{X}^l(M)$ has degree $l-1$. Hence $[\nu,\zeta]$ has degree $k-1+l-1$, and therefore it is an element of $\mathfrak{X}^{k+l-1}(M)$. In contrast with this, we have

$$\wedge : \mathfrak{X}^k(M) \times \mathfrak{X}^l(M) \to \mathfrak{X}^{k+l}(M).$$

The next result follows from the defining properties of the Schouten bracket. Its proof can be found in the appendix.

Lemma 2.2.5. *If $f \in C^\infty(M)$ and $\nu \in \mathfrak{X}^k(M)$, then $[f,\nu] = -\iota_{df}\nu$.*

We also include an explicit definition of the Schouten bracket.

Definition 2.2.6. Let $\nu \in \mathfrak{X}^k(M)$ and $\zeta \in \mathfrak{X}^l(M)$ be multivector fields. Their Schouten bracket is the multivector field $[\nu, \zeta] \in \mathfrak{X}^{k+l-1}(M)$ defined by

$$[\nu, \zeta] = \nu \circ \zeta - (-1)^{(k-1)(l-1)} \zeta \circ \nu,$$

where we set

$$\zeta \circ \nu(df_1, \dots, df_{k+l-1}) := \sum_\sigma sgn(\sigma) \overline{\zeta}(\overline{\nu}(f_{\sigma(1)}, \dots, f_{\sigma(k)}), f_{\sigma(k+1)}, \dots, f_{\sigma(k+l-1)}).$$

The sum is over all $(k, l-1)$-shuffles, that is the $\sigma \in S_{k+l-1}$ satisfying $\sigma(1) < \sigma(2) < \cdots < \sigma(k)$ and $\sigma(k+1) < \sigma(k+2) < \cdots < \sigma(k+l-1)$. And we define $\overline{\nu}(f_1, \dots, f_k) := \nu(df_1, \dots, df_k)$.

Remark 2.2.7. It suffices to define $[\nu, \zeta]$ on exact 1-forms since they locally span $\Omega^1(M)$ and multivector fields are $C^\infty(M)$-multilinear. The explicit definition is of little use in practice; for computations we rather use the defining properties in Theorem 2.2.2.

2.3 Almost Poisson structures (2)

We defined an almost Poisson structure on M in terms of a bilinear bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$. There is a second description, in terms of a bivector field.

Proposition 2.3.1. *For a smooth manifold M, there is a $1:1$ correspondence between almost Poisson brackets $\{\cdot, \cdot\}$ and bivector fields $\Pi \in \mathfrak{X}^2(M)$, given by*

$$\{f, g\} = \Pi(df, dg) = \langle \Pi, df \wedge dg \rangle.$$

Proof. Let $\{\cdot, \cdot\}$ be an almost Poisson bracket on M. We first show that $\{f, g\}(x_0)$ only depends on $d_{x_0} f$ and $d_{x_0} g$. To do this, it suffices to show that $d_{x_0} f = 0$ implies $\{f, g\}(x_0) = 0$.
By Corollary 8.4.2 in the appendix, we can take a coordinate neighborhood U of x_0 on which[1]

$$\begin{aligned} f(x) &= f(x_0) + \sum_{i=1}^n (x^i - x_0^i) \frac{\partial f}{\partial x^i}(x_0) + \sum_{i,j=1}^n (x^i - x_0^i)(x^j - x_0^j) g_{ij}(x) \\ &= f(x_0) + \sum_{i=1}^n (x^i - x_0^i) \frac{\partial f}{\partial x^i}(x_0) + \sum_{i=1}^n (x^i - x_0^i) \sum_{j=1}^n (x^j - x_0^j) g_{ij}(x), \end{aligned}$$

for functions $g_{ij} \in C^\infty(U)$. Since by assumption $d_{x_0} f = 0$, it follows that we can locally write

$$f(x) = f(x_0) + \sum_{i=1}^n (x^i - x_0^i) \sum_{j=1}^n (x^j - x_0^j) g_{ij}(x) = f(x_0) + \sum_{i=1}^n \alpha_i(x) \beta_i(x),$$

where we defined $\alpha_i(x) := x^i - x_0^i$ and $\beta_i(x) := \sum_{j=1}^n (x^j - x_0^j) g_{ij}(x)$. Note that $\alpha_i(x)$ and $\beta_i(x)$ vanish at x_0. Using the Leibniz identity, we get

$$\begin{aligned} \{f, g\}(x_0) &= \Big\{ f(x_0) + \sum_{i=1}^n \alpha_i(x) \beta_i(x), g \Big\}(x_0) \\ &= \{f(x_0), g\}(x_0) + \sum_{i=1}^n \alpha_i(x_0) \{\beta_i, g\}(x_0) + \sum_{i=1}^n \beta_i(x_0) \{\alpha_i, g\}(x_0) \end{aligned}$$

[1] We denote the coordinates $x = (x^1, \dots, x^n)$ by upper indices to avoid double lower indices.

$$= \{f(x_0), g\}(x_0).$$

However, the Leibniz identity also implies that

$$\{1, g\} = \{1 \cdot 1, g\} = \{1, g\} \cdot 1 + 1 \cdot \{1, g\} = 2 \cdot \{1, g\},$$

hence $\{1, g\} = 0$. By linearity, also $f(x_0)\{1, g\} = \{f(x_0), g\} = 0$, from which it follows that $\{f, g\}(x_0) = 0$. Thus we can write

$$\{f, g\}(x) = \Pi_x(d_x f, d_x g), \tag{2.1}$$

for a smooth field of skew-symmetric bilinear maps $\Pi_x : T_x^*M \times T_x^*M \to \mathbb{R}$. Equation (2.1) defines Π uniquely as a $C^\infty(M)$-bilinear, skew-symmetric map $\Pi : \Omega^1(M) \times \Omega^1(M) \to C^\infty(M)$. Conversely, given a bivector field $\Pi \in \mathfrak{X}^2(M)$, we define a bracket $\{\cdot, \cdot\}$ on $C^\infty(M)$ in the prescribed way:

$$\{f, g\} := \Pi(df, dg).$$

It is straightforward to check that this bracket satisfies the necessary properties of Definition 2.1.1. □

Almost Poisson structures are however not exactly the objects of interest for us. Poisson structures are.

2.4 Poisson structures

Definition 2.4.1. An almost Poisson structure $\{\cdot, \cdot\}$ on a manifold M is called a Poisson structure if it satisfies the Jacobi identity, that is

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \qquad \text{for all } f, g, h \in C^\infty(M).$$

A Poisson manifold $(M, \{\cdot, \cdot\})$ is a manifold M equipped with a Poisson structure $\{\cdot, \cdot\}$. The corresponding bivector field Π is called a Poisson tensor.

By Proposition 2.3.1, we know that with a bivector Π comes an almost Poisson bracket $\{\cdot, \cdot\}$. In general, this bracket does not satisfy the Jacobi identity (i.e. it is not Poisson). We will now see which property characterizes bivectors Π that do correspond to Poisson brackets.

Definition 2.4.2. Given an almost Poisson structure $\{\cdot, \cdot\}$ on M, we define the jacobiator J on $C^\infty(M)$ by

$$J(f, g, h) = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\}.$$

One can check that the jacobiator is alternating and a derivation in each of its arguments. By an argument identical to the proof of Proposition 2.3.1, it follows that the jacobiator J corresponds to a trivector field $\mathfrak{J} \in \mathfrak{X}^3(M)$ such that $\mathfrak{J}(df, dg, dh) = J(f, g, h)$.

Proposition 2.4.3. *Let $(M, \{\cdot, \cdot\})$ be almost Poisson. Then $[\Pi, \Pi] = 2\mathfrak{J}$.*

Proof. By definition of the Schouten bracket:

$$\begin{aligned} \frac{1}{2}[\Pi, \Pi](df_1, df_2, df_3) &= \big(\Pi \circ \Pi\big)(df_1, df_2, df_3) \\ &= \sum_\sigma sgn(\sigma)\overline{\Pi}\big(\overline{\Pi}\big(f_{\sigma(1)} f_{\sigma(2)}\big), f_{\sigma(3)}\big), \end{aligned}$$

where the sum is over all $\sigma \in S_3$ with $\sigma(1) < \sigma(2)$. That is, we sum over the permutations (1)(2)(3), (1)(23) and (123). Their signs are +1, −1, +1 respectively. Hence

$$\begin{aligned}\frac{1}{2}[\Pi,\Pi](df_1,df_2,df_3) &= \overline{\Pi}\big(\overline{\Pi}(f_1,f_2),f_3\big)+\overline{\Pi}\big(\overline{\Pi}(f_2,f_3),f_1\big)-\overline{\Pi}\big(\overline{\Pi}(f_1,f_3),f_2\big)\\ &= \overline{\Pi}\big(\Pi(df_1,df_2),f_3\big)+\overline{\Pi}\big(\Pi(df_2,df_3),f_1\big)-\overline{\Pi}\big(\Pi(df_1,df_3),f_2\big)\\ &= \overline{\Pi}\big(\{f_1,f_2\},f_3\big)+\overline{\Pi}\big(\{f_2,f_3\},f_1\big)-\overline{\Pi}\big(\{f_1,f_3\},f_2\big)\\ &= \Pi\big(d\{f_1,f_2\},df_3\big)+\Pi\big(d\{f_2,f_3\},df_1\big)-\Pi\big(d\{f_1,f_3\},df_2\big)\\ &= \{\{f_1,f_2\},f_3\}+\{\{f_2,f_3\},f_1\}+\{\{f_3,f_1\},f_2\} \qquad (2.2)\\ &= J(f_1,f_2,f_3)\\ &= \mathfrak{J}(df_1,df_2,df_3).\end{aligned}$$

Hence $[\Pi,\Pi] = 2\mathfrak{J}$. □

It follows that the Jacobi identity for the bracket $\{\cdot,\cdot\}$ is equivalent with the equation $[\Pi,\Pi] = 0$ for the corresponding bivector field Π.

Corollary 2.4.4. *Let $(M,\{\cdot,\cdot\})$ be a Poisson manifold. Then the associated bivector field $\Pi \in \mathfrak{X}^2(M)$ satisfies $[\Pi,\Pi] = 0$. Conversely, every bivector field $\Pi \in \mathfrak{X}^2(M)$ satisfying this relation defines a Poisson bracket by $\{f,g\} := \Pi(df,dg)$.*

Remark 2.4.5. By Remark 2.2.3, the Schouten bracket of bivector fields is symmetric. Hence, the condition $[\Pi,\Pi] = 0$ is not vacuous.

Example 2.4.6. We reconsider Example 2.1.2. For a finite dimensional Lie algebra $(\mathfrak{g},[\cdot,\cdot])$, we established that

$$\{f,h\}(\xi) = \langle [d_\xi f, d_\xi h],\xi\rangle$$

defines an almost Poisson bracket $\{\cdot,\cdot\}$ on $\mathfrak{g}^*$. In fact, one can check that this bracket satisfies the Jacobi identity. Therefore it is a Poisson structure on $\mathfrak{g}^*$, called the Lie-Poisson bracket.

Example 2.4.7. Let M be a 2-dimensional manifold. Then every bivector Π is a Poisson tensor. Indeed, $[\Pi,\Pi] \in \mathfrak{X}^3(M)$ hence necessarily $[\Pi,\Pi] = 0$.

Example 2.4.8. Quite trivially, any manifold M is Poisson when endowed with the zero bracket $\{\cdot,\cdot\} \equiv 0$.

We finish this section with a few words about coordinate representations of Poisson structures.

Definition 2.4.9. Let $(U, x_1,\dots,x_n)$ be local coordinates on a Poisson manifold $(M,\{\cdot,\cdot\})$ with Poisson bivector Π. Structure functions $\Pi_{i,j} \in C^\infty(U)$ are defined by

$$\Pi_{i,j}(x) = \{x_i,x_j\}(x) = \Pi_x(d_x x_i, d_x x_j).$$

Note that $\Pi_{i,j} = -\Pi_{j,i}$ by skew-symmetry.

In the local basis $\{\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j} : 1 \le i < j \le n\}$ of $\mathfrak{X}^2(M)$, we write $\Pi = \sum_{i<j} h_{i,j}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$ for locally defined functions $h_{i,j}$. Evaluating both sides in (dx_i,dx_j) gives $h_{i,j} = \Pi(dx_i,dx_j) = \Pi_{ij}$. Hence

$$\Pi = \sum_{i<j}\Pi_{i,j}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j} = \frac{1}{2}\sum_{i,j}\Pi_{i,j}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}.$$

Equivalently, we have a local expression for the bracket:

$$\{f,g\} = \Pi(df,dg) = \Pi\left(\sum_i \frac{\partial f}{\partial x_i}dx_i, \sum_j \frac{\partial f}{\partial x_j}dx_j\right) = \sum_{i,j}\Pi(dx_i,dx_j)\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} = \sum_{i,j}\Pi_{i,j}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}.$$

Example 2.4.10. One can derive a coordinate expression for the Lie-Poisson bracket on $\mathfrak{g}^*$ as follows. Let $\{v_1,\ldots,v_n\}$ be a basis of $\mathfrak{g}$ and let $\mu_1,\ldots,\mu_n$ be the coordinate functions on $\mathfrak{g}^*$ corresponding to the dual basis. Introduce structure constants c_{ijk} satisfying

$$[v_i,v_j] = \sum_{k=1}^n c_{ijk}v_k.$$

Then one can check that

$$\{f,g\} = \sum_{i,j,k=1}^n c_{ijk}\mu_k\frac{\partial f}{\partial \mu_i}\frac{\partial g}{\partial \mu_j}. \tag{2.3}$$

2.5 Symplectic versus Poisson structures

Symplectic manifolds are the nicest examples of Poisson manifolds. The aim of this section is showing that symplectic structures correspond to non-degenerate Poisson structures.

Definition 2.5.1. Let $\Pi \in \mathfrak{X}^2(M)$ be a bivector field. It determines a sharp map

$$\Pi^\sharp : \Omega^1(M) \to \mathfrak{X}(M) : \alpha \mapsto \iota_\alpha \Pi.$$

That is, $\langle \beta, \Pi^\sharp(\alpha)\rangle = \langle \alpha\wedge\beta, \Pi\rangle$ for all $\alpha,\beta \in \Omega^1(M)$. The map $\Pi^\sharp$ is $C^\infty(M)$-linear, whence induced by a morphism of vector bundles $\Pi^\sharp : T^*M \to TM$.

Definition 2.5.2. The rank of a bivector $\Pi \in \mathfrak{X}^2(M)$ at a point $x \in M$ is the rank of the linear map $\Pi^\sharp_x : T^*_xM \to T_xM$.

Remark 2.5.3. Let $(x_1,\ldots,x_n)$ be local coordinates around $x \in M$. With respect to the bases $\{d_xx_1,\ldots,d_xx_n\}$ and $\{\frac{\partial}{\partial x_1}|_x,\ldots,\frac{\partial}{\partial x_n}|_x\}$ of T^*_xM resp. T_xM, the matrix of $\Pi^\sharp_x$ is equal to the transpose of the matrix of the bilinear map $\Pi_x : T^*_xM \times T^*_xM \to \mathbb{R}$. That is,

$$\left[\Pi^\sharp_x\right] = \left[\Pi_x(d_xx_i,d_xx_j)\right]^T_{i,j} = \left[\Pi_{i,j}(x)\right]^T_{i,j}.$$

This is true since

$$\begin{aligned}\Pi^\sharp(dx_k) = \iota_{dx_k}\left(\frac{1}{2}\sum_{i,j}\Pi_{i,j}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}\right) &= \frac{1}{2}\sum_{i,j}\Pi_{i,j}\left(\delta_{ik}\frac{\partial}{\partial x_j} - \delta_{jk}\frac{\partial}{\partial x_i}\right)\\ &= \frac{1}{2}\sum_j \Pi_{k,j}\frac{\partial}{\partial x_j} - \frac{1}{2}\sum_i \Pi_{i,k}\frac{\partial}{\partial x_i} = \sum_j \Pi_{k,j}\frac{\partial}{\partial x_j}.\end{aligned}$$

Remark 2.5.4. By skew-symmetry, the rank of Π at any point $x \in M$ is an even number. This is a direct consequence of the Standard Form Theorem for skew-symmetric bilinear maps (Proposition 8.1.1 in the appendix).

Definition 2.5.5. A bivector field $\Pi \in \mathfrak{X}^2(M)$ is called regular if its rank is the same at all points $x \in M$.

Definition 2.5.6. A bivector field $\Pi \in \mathfrak{X}^2(M)$ is called non-degenerate at $x \in M$ if the map $\Pi_x^\sharp : T_x^*M \to T_xM$ is an isomorphism. A bivector field is non-degenerate if it is non-degenerate at all $x \in M$. In this case, the map $\Pi^\sharp : T^*M \to TM$ is a vector bundle isomorphism.

Remark 2.5.7. Assume that $\dim(M) = 2n$. Non-degeneracy of $\Pi \in \mathfrak{X}^2(M)$ is easily checked as follows: Π_x is non-degenerate if and only if $\wedge^n \Pi_x \neq 0$. For a proof of this fact, see Lemma 8.1.2 in the appendix.

The key result in light of the aim of this section, is the following.

Proposition 2.5.8. *There is a* $1:1$ *correspondence between non-degenerate bivector fields* $\Pi \in \mathfrak{X}^2(M)$ *and non-degenerate* 2*-forms* $\omega \in \Omega^2(M)$*, given by*

$$\omega^\flat = -(\Pi^\sharp)^{-1} \longleftrightarrow \Pi^\sharp = -(\omega^\flat)^{-1}$$

Under this correspondence, we have

$$\big[\Pi, \Pi\big](\alpha, \beta, \gamma) = 2d\omega\big(\Pi^\sharp(\alpha), \Pi^\sharp(\beta), \Pi^\sharp(\gamma)\big) \quad \textit{for } \alpha, \beta, \gamma \in \Omega^1(M). \tag{2.4}$$

Proof. The first statement is clear. It is enough to check that equation (2.4) holds on exact 1-forms since these locally span $\Omega^1(M)$ and both sides of (2.4) are $C^\infty(M)$-trilinear.
By Equation (2.2) in Proposition 2.4.3, we have

$$\big[\Pi, \Pi\big]\big(df_1, df_2, df_3\big) = 2\big(\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\}\big).$$

On the other hand, the invariant formula for the exterior derivative gives

$$\begin{aligned} d\omega\big(\Pi^\sharp(df_1), \Pi^\sharp(df_2), \Pi^\sharp(df_3)\big) = {} & \Pi^\sharp(df_1)\Big(\omega\big(\Pi^\sharp(df_2), \Pi^\sharp(df_3)\big)\Big) - \Pi^\sharp(df_2)\Big(\omega\big(\Pi^\sharp(df_1), \Pi^\sharp(df_3)\big)\Big) \\ & + \Pi^\sharp(df_3)\Big(\omega\big(\Pi^\sharp(df_1), \Pi^\sharp(df_2)\big)\Big) - \omega\Big(\big[\Pi^\sharp(df_1), \Pi^\sharp(df_2)\big], \Pi^\sharp(df_3)\Big) \\ & + \omega\Big(\big[\Pi^\sharp(df_1), \Pi^\sharp(df_3)\big], \Pi^\sharp(df_2)\Big) - \omega\Big(\big[\Pi^\sharp(df_2), \Pi^\sharp(df_3)\big], \Pi^\sharp(df_1)\Big). \end{aligned}$$

Here,

$$\begin{aligned} \Pi^\sharp(df_1)\Big(\omega\big(\Pi^\sharp(df_2), \Pi^\sharp(df_3)\big)\Big) &= \Pi^\sharp(df_1)\Big(\omega^\flat\big(\Pi^\sharp(df_2)\big)\big(\Pi^\sharp(df_3)\big)\Big) \\ &= -\Pi^\sharp(df_1)\big(df_2\big(\Pi^\sharp(df_3)\big)\big) && (\text{since } \omega^\flat = -(\Pi^\sharp)^{-1}) \\ &= -\Pi^\sharp(df_1)\big(\Pi^\sharp(df_3)(f_2)\big) \\ &= -\Pi^\sharp(df_1)\big(\{f_3, f_2\}\big) && (\text{since } \Pi^\sharp(df_3) = \Pi(df_3, \cdot) = \{f_3, \cdot\}) \\ &= -\{f_1, \{f_3, f_2\}\}. \end{aligned}$$

Similarly,

$$\Pi^\sharp(df_2)\Big(\omega\big(\Pi^\sharp(df_1), \Pi^\sharp(df_3)\big)\Big) = -\{f_2, \{f_3, f_1\}\}$$

and

$$\Pi^\sharp(df_3)\Big(\omega\big(\Pi^\sharp(df_1), \Pi^\sharp(df_2)\big)\Big) = -\{f_3, \{f_2, f_1\}\}.$$

Next,

$$\begin{aligned} \omega\Big(\big[\Pi^\sharp(df_1), \Pi^\sharp(df_2)\big], \Pi^\sharp(df_3)\Big) &= -\omega^\flat\big(\Pi^\sharp(df_3)\big)\Big(\big[\Pi^\sharp(df_1), \Pi^\sharp(df_2)\big]\Big) \\ &= df_3\Big(\big[\Pi^\sharp(df_1), \Pi^\sharp(df_2)\big]\Big) && (\text{since } \omega^\flat = -(\Pi^\sharp)^{-1}) \end{aligned}$$

$$
\begin{aligned}
&= \big[\Pi^\sharp(df_1), \Pi^\sharp(df_2)\big](f_3) \\
&= \Pi^\sharp(df_1)\left(\Pi^\sharp(df_2)(f_3)\right) - \Pi^\sharp(df_2)\left(\Pi^\sharp(df_1)(f_3)\right) \\
&= \{f_1, \{f_2, f_3\}\} - \{f_2, \{f_1, f_3\}\}.
\end{aligned}
$$

Similarly,

$$\omega\Big(\big[\Pi^\sharp(df_1), \Pi^\sharp(df_3)\big], \Pi^\sharp(df_2)\Big) = \{f_1, \{f_3, f_2\}\} - \{f_3, \{f_1, f_2\}\}$$

and

$$\omega\Big(\big[\Pi^\sharp(df_2), \Pi^\sharp(df_3)\big], \Pi^\sharp(df_1)\Big) = \{f_2, \{f_3, f_1\}\} - \{f_3, \{f_2, f_1\}\}.$$

It follows that

$$
\begin{aligned}
d\omega\big(\Pi^\sharp(df_1), \Pi^\sharp(df_2), \Pi^\sharp(df_3)\big) &= -\{f_1, \{f_3, f_2\}\} + \{f_2, \{f_3, f_1\}\} - \{f_3, \{f_2, f_1\}\} \\
&\quad - \{f_1, \{f_2, f_3\}\} + \{f_2, \{f_1, f_3\}\} + \{f_1, \{f_3, f_2\}\} \\
&\quad - \{f_3, \{f_1, f_2\}\} - \{f_2, \{f_3, f_1\}\} + \{f_3, \{f_2, f_1\}\} \\
&= \{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\}.
\end{aligned}
$$

Thus,

$$\big[\Pi, \Pi\big](df_1, df_2, df_3) = 2d\omega\big(\Pi^\sharp(df_1), \Pi^\sharp(df_2), \Pi^\sharp(df_3)\big).$$

□

We conclude:

Corollary 2.5.9. *On a manifold M, there is a $1:1$ correspondence between non-degenerate Poisson structures and symplectic stuctures.*

Proof. If $\Pi \in \mathfrak{X}^2(M)$ is non-degenerate, then $\Pi^\sharp : \Omega^1(M) \to \mathfrak{X}(M)$ is an isomorphism (of $C^\infty(M)$-modules), in which case $d\omega = 0$ if and only if $d\omega\big(\Pi^\sharp(\alpha), \Pi^\sharp(\beta), \Pi^\sharp(\gamma)\big) = 0$ for all $\alpha, \beta, \gamma \in \Omega^1(M)$. It follows that the correspondence of Proposition 2.5.8 matches non-degenerate bivectors Π satisfying $[\Pi, \Pi] = 0$ with non-degenerate 2-forms ω satisfying $d\omega = 0$. □

In local coordinates, we make the transition between the Poisson bivector Π and its associated symplectic form ω as follows. Choose local frames $\{dx_1, \dots, dx_n\}$ and $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ of T^*M resp. TM. Write $\Pi = \sum_{i<j} \Pi_{i,j} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. By non-degeneracy, the matrix $[\Pi] = [\Pi_{i,j}]_{i,j}$ is invertible, and

$$[\omega] = [\omega^\flat]^T = \left(-[\Pi^\sharp]^{-1}\right)^T = -\left([\Pi^\sharp]^T\right)^{-1} = -[\Pi]^{-1}$$

It follows that $\omega = \sum_{i<j} \omega_{i,j} dx_i \wedge dx_j$, where $[\omega_{i,j}]_{i,j} = -[\Pi_{i,j}]^{-1}$.

Example 2.5.10. The canonical Poisson structure on $\mathbb{R}^{2n}$ with coordinates $(q_1, p_1, \dots, q_n, p_n)$ is $\Pi = \sum_{i=1}^n \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}$. It is non-degenerate, and corresponds to the canonical symplectic form $\omega = -\Pi^{-1} = \sum_{i=1}^n dq_i \wedge dp_i$.

2.6 Poisson maps

Having defined the objects of the Poisson category (namely, Poisson manifolds), we now address the morphisms of this category.

Definition 2.6.1. A smooth map $\Phi : (M, \{\cdot,\cdot\}_M) \to (N, \{\cdot,\cdot\}_N)$ between Poisson manifolds is called a Poisson map when

$$\Phi^*\left(\{f, g\}_N\right) = \{\Phi^*(f), \Phi^*(g)\}_M \quad \text{for all } f, g \in C^\infty(N).$$

That is, $\{f \circ \Phi, g \circ \Phi\}_M = \{f, g\}_N \circ \Phi$ for all $f, g \in C^\infty(N)$.

Remark 2.6.2. If $(M, \{\cdot,\cdot\})$ is a Poisson manifold, then $(C^\infty(M), \{\cdot,\cdot\})$ is a Poisson algebra, i.e. a commutative, associative algebra with a Lie algebra structure satisfying the Leibniz identity. Definition 2.6.1 states that Poisson maps are those maps $\Phi : (M, \{\cdot,\cdot\}_M) \to (N, \{\cdot,\cdot\}_N)$ whose pullback $\Phi^* : (C^\infty(N), \{\cdot,\cdot\}_N) \to (C^\infty(M), \{\cdot,\cdot\}_M)$ is a morphism of Poisson algebras.

We now give a characterization of Poisson maps in terms of the Poisson bivectors.

Lemma 2.6.3. *Let $(M, \{\cdot,\cdot\}_M)$ and $(N, \{\cdot,\cdot\}_N)$ be Poisson manifolds. Denote by $\Pi_M \in \mathfrak{X}^2(M)$ and $\Pi_N \in \mathfrak{X}^2(N)$ the corresponding Poisson bivectors. A smooth map $\Phi : M \to N$ is Poisson if and only if Π_N and Π_M are Φ-related.*

Proof. First assume that Π_N and Π_M are Φ-related. For $f, g \in C^\infty(N)$, we get

$$\begin{aligned}
\{f \circ \Phi, g \circ \Phi\}_M(p) &= (\Pi_M)_p\big(d_p(f \circ \Phi), d_p(g \circ \Phi)\big) \\
&= (\Pi_M)_p\big(d_{\Phi(p)} f \circ d_p\Phi, d_{\Phi(p)} g \circ d_p\Phi\big) && \text{(chain rule)} \\
&= (\Pi_M)_p\big(\Phi^*(d_{\Phi(p)} f), \Phi^*(d_{\Phi(p)} g)\big) \\
&= (\Pi_N)_{\Phi(p)}\big(d_{\Phi(p)} f, d_{\Phi(p)} g\big) && \text{(Remark 2.2.1)} \\
&= \{f, g\}_N(\Phi(p)).
\end{aligned}$$

This shows that $\{f \circ \Phi, g \circ \Phi\}_M = \{f, g\}_N \circ \Phi$, hence Φ is Poisson.
Conversely, assume that Φ is a Poisson map. We have to show that $(\Pi_N)_{\Phi(p)} = (d_p\Phi)(\Pi_M)_p$ for all $p \in M$. It suffices to prove this equality on differentials of functions. We have

$$\begin{aligned}
(d_p\Phi)(\Pi_M)_p\big(d_{\Phi(p)} f, d_{\Phi(p)} g\big) &= (\Pi_M)_p\big(\Phi^*(d_{\Phi(p)} f), \Phi^*(d_{\Phi(p)} g)\big) && \text{(Remark 2.2.1)} \\
&= (\Pi_M)_p\big(d_{\Phi(p)} f \circ d_p\Phi, d_{\Phi(p)} g \circ d_p\Phi\big) \\
&= (\Pi_M)_p\big(d_p(f \circ \Phi), d_p(g \circ \Phi)\big) && \text{(chain rule)} \\
&= \{f \circ \Phi, g \circ \Phi\}_M(p) \\
&= \{f, g\}_N(\Phi(p)) && \text{(since } \Phi \text{ is Poisson)} \\
&= (\Pi_N)_{\Phi(p)}\big(d_{\Phi(p)} f, d_{\Phi(p)} g\big).
\end{aligned}$$

This shows that Π_N and Π_M are Φ-related. □

Example 2.6.4. Let $(M_1, \{\cdot,\cdot\}_1)$ and $(M_2, \{\cdot,\cdot\}_2)$ be Poisson manifolds. Their direct product $M_1 \times M_2$ can be equipped with the natural bracket

$$\{f(x_1, x_2), g(x_1, x_2)\} := \{f_{x_2}, g_{x_2}\}_1(x_1) + \{f_{x_1}, g_{x_1}\}_2(x_2),$$

where we use the notation $h_{x_1}(x_2) = h_{x_2}(x_1) = h(x_1, x_2)$ for $h \in C^\infty(M_1 \times M_2), x_1 \in M_1$ and $x_2 \in M_2$. This is a Poisson bracket on $M_1 \times M_2$, called the product Poisson structure. With respect to this product Poisson structure, the projection maps $M_1 \times M_2 \to M_1$ and $M_1 \times M_2 \to M_2$ are Poisson maps.

One last characterization of Poisson maps involves the sharp maps associated with the Poisson bivectors. It will be useful in the next section.

Lemma 2.6.5. *Let (M, Π_M) and (N, Π_N) be Poisson manifolds. A smooth map $\Phi : M \to N$ is Poisson if and only if the following diagram commutes for all $x \in M$:*

$$\begin{array}{ccc} T_x^*M & \xrightarrow{(\Pi_M^\sharp)_x} & T_xM \\ {\scriptstyle (d_x\Phi)^*}\big\uparrow & & \big\downarrow{\scriptstyle d_x\Phi} \\ T^*_{\Phi(x)}N & \xrightarrow{(\Pi_N^\sharp)_{\Phi(x)}} & T_{\Phi(x)}N \end{array}$$

Proof. It is enough to check this assertion on differentials of functions. Let $f \in C^\infty(N)$. We will show that the actions of $\big(d_x\Phi \circ (\Pi_M^\sharp)_x \circ (d_x\Phi)^*\big)(d_{\Phi(x)}f)$ and $\big(\Pi_N^\sharp\big)_{\Phi(x)}(d_{\Phi(x)}f)$ coincide on any function $g \in C^\infty(N)$ if and only if Φ is Poisson. We have

$$\begin{aligned}
\Big[\big(d_x\Phi \circ (\Pi_M^\sharp)_x \circ (d_x\Phi)^*\big)(d_{\Phi(x)}f)\Big](g) &= \Big[\big(d_x\Phi \circ (\Pi_M^\sharp)_x\big)(d_{\Phi(x)}f \circ d_x\Phi)\Big](g) \\
&= \Big[\big(d_x\Phi \circ (\Pi_M^\sharp)_x\big)\big(d_x(f \circ \Phi)\big)\Big](g) \\
&= \Big[d_x\Phi\big(\{f \circ \Phi, \bullet\}_M(x)\big)\Big](g) \\
&= d_{\Phi(x)}g\big(d_x\Phi\big(\{f \circ \Phi, \bullet\}_M(x)\big)\big) \\
&= d_x(g \circ \Phi)\big(\{f \circ \Phi, \bullet\}_M(x)\big) \\
&= \{f \circ \Phi, g \circ \Phi\}_M(x),
\end{aligned}$$

whereas

$$\big[(\Pi_N^\sharp)_{\Phi(x)}(d_{\Phi(x)}f)\big](g) = \{f, g\}_N(\Phi(x)).$$

The claim follows. ☐

Remark 2.6.6. Let (M, ω_M) and (N, ω_N) be symplectic manifolds. Then asking for a map $\Phi : M \to N$ to be symplectic (i.e. $\Phi^*\omega_N = \omega_M$) is not the same as asking for Φ to be Poisson. For instance, consider

$$i : (\mathbb{R}^2, p_1, q_1) \to (\mathbb{R}^4, p_1, q_1, p_2, q_2) : (p_1, q_1) \mapsto (p_1, q_1, 0, 0).$$

Here $\mathbb{R}^2$ and $\mathbb{R}^4$ are endowed with their respective canonical Poisson structures $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$. The corresponding symplectic forms are $\omega_1 = dp_1 \wedge dq_1$ and $\omega_2 = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, respectively. Then i is symplectic, but not Poisson since

$$\{p_2 \circ i, q_2 \circ i\}_1 = \{0, 0\}_1 = 0,$$

whereas

$$\{p_2, q_2\}_2 \circ i = 1 \circ i = 1.$$

Remark 2.6.7. A symplectic realization of a Poisson manifold N is a Poisson map $\Phi : M \to N$, where M is a symplectic manifold. One can show that every Poisson manifold admits a surjective submersive symplectic realization.

2.7 Poisson vector fields

Definition 2.7.1. Let $(M,\{\cdot,\cdot\})$ be a Poisson manifold. For any $f\in C^\infty(M)$ we have a linear map

$$X_f : C^\infty(M)\to C^\infty(M) : h\mapsto X_f(h)=\{f,h\}.$$

The Leibniz identity of $\{\cdot,\cdot\}$ says that X_f is a derivation. It thus corresponds to a vector field, called the hamiltonian vector field of the function f.

Remark 2.7.2. If $\Pi\in\mathfrak{X}^2(M)$ is the Poisson bivector corresponding to the bracket $\{\cdot,\cdot\}$, then we can write

$$X_f=\{f,\bullet\}=\iota_{df}\Pi=\Pi^\sharp(df).$$

The assignment $C^\infty(M)\to\mathfrak{X}(M) : f\mapsto X_f$ is a morphism of Lie algebras:

Lemma 2.7.3. *Let $(M,\{\cdot,\cdot\})$ be a Poisson manifold. Then*

$$[X_f,X_g]=X_{\{f,g\}}.$$

Proof. For $h\in C^\infty(M)$, we have

$$\begin{aligned}\left([X_f,X_g]-X_{\{f,g\}}\right)h &= X_fX_gh-X_gX_fh-X_{\{f,g\}}h\\ &= X_f\{g,h\}-X_g\{f,h\}-\{\{f,g\},h\}\\ &= \{f,\{g,h\}\}-\{g,\{f,h\}\}-\{\{f,g\},h\}\\ &= -\{\{f,g\},h\}-\{\{g,h\},f\}-\{\{h,f\},g\}\\ &= 0,\end{aligned}$$

where we used skew-symmetry of $\{\cdot,\cdot\}$ and the Jacobi identity. □

Hamiltonian vector fields are expressed in local coordinates as follows.

Lemma 2.7.4. *Let $(M,\{\cdot,\cdot\})$ be a Poisson manifold with local coordinates $(U,x_1,\dots,x_n)$. Then for all $f\in C^\infty(M)$:*

$$X_f|_U=\sum_{i,j=1}^n\{x_i,x_j\}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}=\sum_{j=1}^n\{f,x_j\}\frac{\partial}{\partial x_j}.$$

Proof. Writing

$$\Pi=\frac{1}{2}\sum_{i,j=1}^n\{x_i,x_j\}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$$

for the associated Poisson bivector, we get on U that

$$\begin{aligned}X_f=\iota_{df}\Pi &= \frac{1}{2}\sum_{i,j=1}^n\{x_i,x_j\}\left(\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}-\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}\right)\\ &= \frac{1}{2}\sum_{i,j=1}^n\{x_i,x_j\}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}-\frac{1}{2}\sum_{i,j=1}^n\{x_i,x_j\}\frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}\\ &= \sum_{i,j=1}^n\{x_i,x_j\}\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j},\end{aligned}$$

where the last equality is obtained by re-indexing $i \leftrightarrow j$ in the second summation and using skew-symmetry of $\{\cdot,\cdot\}$. Noting that

$$\{f, x_j\} = \Pi(df, dx_j) = \Pi\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i, dx_j\right) = \sum_{i=1}^{n} \Pi(dx_i, dx_j)\frac{\partial f}{\partial x_i} = \sum_{i=1}^{n}\{x_i, x_j\}\frac{\partial f}{\partial x_i}$$

finishes the proof. □

Poisson vector fields are the infinitesimal automorphisms of the Poisson structure.

Definition 2.7.5. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold with Poisson bivector Π. A vector field $X \in \mathfrak{X}(M)$ is a Poisson vector field if the following equivalent conditions hold:

1. $\pounds_X \Pi = 0$;
2. $X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\}$ for all $f, g \in C^\infty(M)$;
3. The flow $\{\phi_t\}$ of X consists of local Poisson diffeomorphisms.

Proof. We have

$$\begin{aligned}
\pounds_X\{f,g\} &= \pounds_X\langle \Pi, df\wedge dg\rangle = \langle \pounds_X \Pi, df\wedge dg\rangle + \langle \Pi, \pounds_X(df\wedge dg)\rangle && \text{(Leibniz rule for pairing)}\\
&= \langle \pounds_X \Pi, df\wedge dg\rangle + \langle \Pi, (\pounds_X df)\wedge dg\rangle + \langle \Pi, df\wedge(\pounds_X dg)\rangle\\
&= \langle \pounds_X \Pi, df\wedge dg\rangle + \langle \Pi, d\pounds_X f\wedge dg\rangle + \langle \Pi, df\wedge d\pounds_X g\rangle && (d \text{ and } \pounds_X \text{ commute})\\
&= \langle \pounds_X \Pi, df\wedge dg\rangle + \langle \Pi, dX(f)\wedge dg\rangle + \langle \Pi, df\wedge dX(g)\rangle.
\end{aligned}$$

Hence

$$X(\{f,g\}) = (\pounds_X\Pi)(df, dg) + \{X(f), g\} + \{f, X(g)\}$$

or

$$(\pounds_X\Pi)(df, dg) = X(\{f,g\}) - \{X(f), g\} - \{f, X(g)\}.$$

This proves that 1. ⇔ 2.
Next, recall the classical formula (see Lemma 8.2.3 in the appendix)

$$\frac{d}{dt}\rho_t^*\omega_t = \rho_t^*\left(\pounds_{v_t}\omega_t + \frac{d\omega_t}{dt}\right).$$

Here $\{\omega_t\}$ is a smooth family of differential k-forms, and $\{\rho_t\}$ is an isotopy with corresponding time dependent vector field $\{v_t\}$. In case $\{\phi_t\}$ is the flow of the vector field X, we get

$$\begin{aligned}
\frac{d}{dt}\big(\{f\circ\phi_t, g\circ\phi_t\}\circ\phi_{-t}\big) &= \frac{d}{dt}\phi_{-t}^*\{\phi_t^* f, \phi_t^* g\} = -\phi_{-t}^*\left(X\{\phi_t^* f, \phi_t^* g\} - \frac{d}{dt}\{\phi_t^* f, \phi_t^* g\}\right)\\
&= -\phi_{-t}^*\left(X\{\phi_t^* f, \phi_t^* g\}\right) + \phi_{-t}^*\left(\{\frac{d}{dt}\phi_t^* f, \phi_t^* g\} + \{\phi_t^* f, \frac{d}{dt}\phi_t^* g\}\right)\\
&= -\phi_{-t}^*\left(X\{\phi_t^* f, \phi_t^* g\}\right) + \phi_{-t}^*\left(\{\phi_t^* X(f), \phi_t^* g\} + \{\phi_t^* f, \phi_t^* X(g)\}\right)\\
&= -\phi_{-t}^*\left(X\{\phi_t^* f, \phi_t^* g\}\right) + \phi_{-t}^*\left(\{X(\phi_t^* f), \phi_t^* g\} + \{\phi_t^* f, X(\phi_t^* g)\}\right).
\end{aligned} \tag{2.5}$$

If $\{\phi_t\}$ consists of Poisson diffeomorphisms, then

$$\frac{d}{dt}\big(\{f\circ\phi_t, g\circ\phi_t\}\circ\phi_{-t}\big) = \frac{d}{dt}\{f, g\} = 0,$$

hence the right hand side of Equation (2.5) is zero for all times t. As ϕ_0 is the identity map, we obtain for $t = 0$ that

$$X(\{f, g\}) = \{X(f), g\} + \{f, X(g)\}.$$

Conversely, if 2. holds then the right hand side of Equation (2.5) is zero. This implies that

$$\{f \circ \phi_t, g \circ \phi_t\} \circ \phi_{-t} = \{f \circ \phi_0, g \circ \phi_0\} \circ \phi_0 = \{f, g\},$$

that is, $\{\phi_t\}$ consists of Poisson diffeomorphisms. This proves that 2. ⇔ 3. □

Remark 2.7.6. The second characterization in Definition 2.7.5 above says that Poisson vector fields are the derivations of the Poisson algebra $(C^\infty(M), \{\cdot, \cdot\})$, both with respect to $\cdot$ and to $\{\cdot, \cdot\}$.

Remark 2.7.7. Hamiltonian vector fields are Poisson. Indeed:

$$\begin{aligned} X_h(\{f, g\}) - \{X_h(f), g\} - \{f, X_h(g)\} &= \{h, \{f, g\}\} - \{\{h, f\}, g\} - \{f, \{h, g\}\} \\ &= -\{\{f, g\}, h\} - \{\{g, h\}, f\} - \{\{h, f\}, g\} \\ &= 0 \qquad\qquad \text{(Jacobi identity).} \end{aligned}$$

The converse is not true in general, not even locally. For instance, if the Poisson structure is identically zero, then every vector field is Poisson while the zero vector field is the only hamiltonian vector field.

2.8 Poisson cohomology

Poisson manifolds have a cohomology theory of their own: Poisson cohomology. We will discuss this invariant and address its relation with de Rham cohomology.

Lemma 2.8.1. *Let (M, Π) be a Poisson manifold. Then for any multivector field $\xi \in \mathfrak{X}^k(M)$, we have*

$$[\Pi, [\Pi, \xi]] = 0.$$

Proof. By the graded Jacobi identity in $\big(\mathfrak{X}^{\bullet-1}(M), [\cdot, \cdot]\big)$ (see Remark 2.2.3), we have

$$-[[\Pi, \xi], \Pi] + (-1)^{k-1}[[\xi, \Pi], \Pi] + (-1)^{k-1}[[\Pi, \Pi], \xi] = 0. \tag{2.6}$$

Since Π is Poisson, $[\Pi, \Pi] = 0$. Graded skew-symmetry of $[\cdot, \cdot]$ gives $[\xi, \Pi] = -(-1)^{k-1}[\Pi, \xi]$. Hence Equation (2.6) becomes

$$-2[[\Pi, \xi], \Pi] = 0.$$

As $[\Pi, [\Pi, \xi]]$ equals $[[\Pi, \xi], \Pi]$ up to sign, it follows that $[\Pi, [\Pi, \xi]] = 0$. □

Definition 2.8.2. Let (M, Π) be a Poisson manifold. Denote by $d_\Pi : \mathfrak{X}^\bullet(M) \to \mathfrak{X}^{\bullet+1}(M)$ the $\mathbb{R}$-linear operator defined by

$$d_\Pi(\xi) = [\Pi, \xi].$$

Lemma 2.8.1 states that d_Π is a differential, that is $d_\Pi \circ d_\Pi = 0$. We get a cochain complex $(\mathfrak{X}^\bullet(M), d_\Pi)$:

$$\cdots \xrightarrow{d_\Pi} \mathfrak{X}^{k-1}(M) \xrightarrow{d_\Pi} \mathfrak{X}^k(M) \xrightarrow{d_\Pi} \mathfrak{X}^{k+1}(M) \xrightarrow{d_\Pi} \cdots,$$

called the Lichnerowicz complex. The cohomology of this complex is the Poisson cohomology. That is, the Poisson cohomology groups are

$$H^k_\Pi(M) := \frac{\operatorname{Ker}\big(d_\Pi : \mathfrak{X}^k(M) \to \mathfrak{X}^{k+1}(M)\big)}{\operatorname{Im}\big(d_\Pi : \mathfrak{X}^{k-1}(M) \to \mathfrak{X}^k(M)\big)}.$$

Remark 2.8.3. The low-dimensional Poisson cohomology groups have an easy interpretation. Using Lemma 2.2.5, we see that $[\Pi, f] = -\iota_{df}\Pi = -X_f$. Hence

$$H^0_\Pi(M) = \{f \in C^\infty(M) : [\Pi, f] = 0\} = \{f \in C^\infty(M) : X_f = 0\},$$

so $H^0_\Pi(M)$ is the space of the so-called Casimir functions.
Next, noting that $[\Pi, X] = -\mathcal{L}_X\Pi$, we obtain for the first Poisson cohomology that

$$H^1_\Pi(M) = \frac{\{X \in \mathfrak{X}(M) : [\Pi, X] = 0\}}{\{[\Pi, f] : f \in C^\infty(M)\}} = \frac{\{X \in \mathfrak{X}(M) : \mathcal{L}_X\Pi = 0\}}{\{X_{-f} : f \in C^\infty(M)\}} = \frac{\{\text{Poisson vector fields}\}}{\{\text{Hamiltonian vector fields}\}}.$$

The second Poisson cohomology group is by definition

$$H^2_\Pi(M) = \frac{\{\Lambda \in \mathfrak{X}^2(M) : [\Pi, \Lambda] = 0\}}{\{[\Pi, Y] : Y \in \mathfrak{X}(M)\}}.$$

To find an interpretation of $H^2_\Pi(M)$, we consider a formal one-parameter deformation of Π by

$$\Pi(\epsilon) = \Pi + \epsilon\Pi_1 + \epsilon^2\Pi_2 + \cdots,$$

for $\Pi_i \in \mathfrak{X}^2(M)$ and ϵ a formal infinitesimal parameter. The condition for $\Pi(\epsilon)$ to be a Poisson bivector is

$$0 = [\Pi(\epsilon), \Pi(\epsilon)] = [\Pi, \Pi] + 2\epsilon[\Pi, \Pi_1] + \epsilon^2\left(2[\Pi, \Pi_2] + [\Pi_1, \Pi_1]\right) + \cdots$$

Since Π is Poisson, we have $[\Pi, \Pi] = 0$. If $[\Pi, \Pi_1] = 0$, then the bivector $\Pi + \epsilon\Pi_1$ satisfies the Jacobi identity up to order ϵ^2:

$$[\Pi + \epsilon\Pi_1, \Pi + \epsilon\Pi_1] = 0 + \mathcal{O}(\epsilon^2).$$

We then call Π_1 an infinitesimal deformation of Π. In case $\Pi_1 = [\Pi, Y] = -\mathcal{L}_Y\Pi$ for some $Y \in \mathfrak{X}(M)$, then Π_1 is called a trivial infinitesimal deformation of Π. This terminology is motivated by the following observation. Let $\varphi_{-\epsilon}$ denote the time $-\epsilon$ flow of $-Y$. Then the pushforwards $(\varphi_{-\epsilon})_*\Pi$ are again Poisson structures, since by Lemma 8.3.2

$$\left[(\varphi_{-\epsilon})_*\Pi, (\varphi_{-\epsilon})_*\Pi\right] = (\varphi_{-\epsilon})_*[\Pi, \Pi] = 0.$$

Moreover,

$$\frac{d}{d\epsilon}(\varphi_{-\epsilon})_*\Pi\bigg|_{\epsilon=0} = \mathcal{L}_{-Y}\Pi = \Pi_1,$$

so that we have an expansion

$$(\varphi_{-\epsilon})_*\Pi = \Pi + \epsilon\Pi_1 + \mathcal{O}(\epsilon^2).$$

We now see that the infinitesimal deformation Π_1 is trivial in the sense that the Poisson structures $(\varphi_{-\epsilon})_*\Pi$ are essentially the same, only expressed in different coordinates. We conclude

$$H^2_\Pi(M) = \frac{\{\text{Infinitesimal deformations of }\Pi\}}{\{\text{Trivial infinitesimal deformations of }\Pi\}}.$$

Heuristically, $H^2_\Pi(M) = T_\Pi\mathcal{M}$ is the "tangent space" at Π to the moduli space of Poisson structures on M, which is obtained by factoring out diffeomorphic Poisson structures:

$$\mathcal{M} = \frac{\text{Poiss}(M)}{\text{Diff}(M)}.$$

Recall that a Poisson tensor Π induces a bundle map $\Pi^\sharp : T^*M \to TM$. By taking exterior powers, we extend it to a map $\wedge^k T^*M \to \wedge^k TM$. On the level of sections, this is a $C^\infty(M)$-linear map, given by

$$\Omega^k(M) \to \mathfrak{X}^k(M) : \alpha_1 \wedge \cdots \wedge \alpha_k \mapsto \Pi^\sharp(\alpha_1) \wedge \cdots \wedge \Pi^\sharp(\alpha_k).$$

We will denote this map by $\Pi^\sharp$ as well. By convention, $\Pi^\sharp(f) = f$ for all $f \in C^\infty(M) = \Omega^0(M)$.

Lemma 2.8.4. *Up to sign, the map $\Pi^\sharp : \Omega^\bullet(M) \to \mathfrak{X}^\bullet(M)$ is a chain map between the de Rham complex*

$$\cdots \xrightarrow{d} \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \cdots$$

and the Lichnerowicz complex

$$\cdots \xrightarrow{d_\Pi} \mathfrak{X}^{k-1}(M) \xrightarrow{d_\Pi} \mathfrak{X}^k(M) \xrightarrow{d_\Pi} \mathfrak{X}^{k+1}(M) \xrightarrow{d_\Pi} \cdots .$$

That is, $\Pi^\sharp(d\eta) = -d_\Pi(\Pi^\sharp(\eta))$ for all $\eta \in \Omega^k(M)$.

Proof. By induction on the degree k of η.
If $\eta \in C^\infty(M)$, then

$$-d_\Pi(\Pi^\sharp(\eta)) = -d_\Pi(\eta) = -[\Pi, \eta] = \iota_{d\eta}\Pi = \Pi^\sharp(d\eta).$$

If $\eta = df$ is an exact 1-form, then

$$\Pi^\sharp(d\eta) = \Pi^\sharp(d^2 f) = 0,$$

and

$$d_\Pi(\Pi^\sharp(\eta)) = d_\Pi(X_f) = [\Pi, X_f] = -\pounds_{X_f}\Pi = 0,$$

since hamiltonian vector fields are Poisson.
If the formula holds for $\eta \in \Omega^p(M)$ and $\mu \in \Omega^q(M)$, then it also holds for $\eta \wedge \mu$. Indeed,

$$\begin{aligned}\Pi^\sharp\big(d(\eta \wedge \mu)\big) &= \Pi^\sharp\big(d\eta \wedge \mu + (-1)^p \eta \wedge d\mu\big) = \Pi^\sharp(d\eta) \wedge \Pi^\sharp(\mu) + (-1)^p \Pi^\sharp(\eta) \wedge \Pi^\sharp(d\mu) \\ &= -d_\Pi\big(\Pi^\sharp(\eta)\big) \wedge \Pi^\sharp(\mu) - (-1)^p \Pi^\sharp(\eta) \wedge d_\Pi\big(\Pi^\sharp(\mu)\big) \\ &= -[\Pi, \Pi^\sharp(\eta)] \wedge \Pi^\sharp(\mu) - (-1)^p \Pi^\sharp(\eta) \wedge [\Pi, \Pi^\sharp(\mu)] \\ &= -[\Pi, \Pi^\sharp(\eta) \wedge \Pi^\sharp(\mu)] \\ &= -d_\Pi\big(\Pi^\sharp(\eta \wedge \mu)\big).\end{aligned}$$

□

Corollary 2.8.5. *We have an induced morphism between cohomology groups*

$$[\Pi^\sharp] : H^k_{dR}(M) \to H^k_\Pi(M) : [\eta] \mapsto [\Pi^\sharp(\eta)].$$

There are some algebraic topological tools for computing Poisson cohomology, one of which is the Mayer-Vietoris sequence. However, explicit computation of Poisson cohomology remains a hard problem. Poisson cohomology groups are generically very big and infinite-dimensional, which is in contrast with de Rham cohomology (for instance, de Rham cohomology groups of a compact manifold are finite-dimensional).

Example 2.8.6. If M is equipped with the zero Poisson structure $\Pi \equiv 0$, then

$$H^0_\Pi(M) = C^\infty(M) \qquad \text{and} \qquad H^1_\Pi(M) = \mathfrak{X}(M).$$

In particular, $H^0_\Pi(M)$ and $H^1_\Pi(M)$ are infinite dimensional (as vector spaces over $\mathbb{R}$).

Example 2.8.7. If (M,Π) is symplectic, then $\Pi^\sharp : T^*M \to TM$ is an isomorphism of vector bundles. Hence the same holds for its exterior powers $\Pi^\sharp : \wedge^k T^*M \to \wedge^k TM$. It follows that on the chain level, we get isomorphisms of $C^\infty(M)$-modules $\Pi^\sharp : \Omega^k(M) \to \mathfrak{X}^k(M)$. Since passing to cohomology is functorial, it follows that the induced maps on cohomology

$$[\Pi^\sharp] : H^k_{dR}(M) \to H^k_\Pi(M)$$

are isomorphisms. Hence for a symplectic manifold, the Poisson cohomology groups are isomorphic to the de Rham cohomology groups.

2.9 Modular vector fields

We dedicate this section to a specific kind of Poisson vector fields, called modular vector fields. They will play an important role in this thesis.

Definition 2.9.1. Let (M^n,Π) be an orientable Poisson manifold. Fix a volume form Ω on M. The modular vector field X^Ω_Π is the derivation given by the map

$$X^\Omega_\Pi : C^\infty(M) \to C^\infty(M) : f \mapsto \frac{\pounds_{X_f}\Omega}{\Omega}. \tag{2.7}$$

Remark 2.9.2. Since $\wedge^n T^*M$ is a line bundle, $\pounds_{X_f}\Omega$ and Ω differ by a factor in $C^\infty(M)$. Hence, the expression on the right in (2.7) is indeed a smooth function.

So the modular vector field measures to what extent Hamiltonian vector fields preserve a given volume form. Let us first check that the definition makes sense.

Lemma 2.9.3. *The assignment* $f \mapsto \frac{\pounds_{X_f}\Omega}{\Omega}$ *is a derivation of* $C^\infty(M)$.

Proof. Since linearity is clear, we only check the Leibniz rule. Let $f,g,h \in C^\infty(M)$. The Leibniz rule for the Poisson bracket $\{\cdot,\cdot\}$ implies that

$$X_{fg}(h) = \{fg,h\} = f\{g,h\} + g\{f,h\} = fX_g(h) + gX_f(h),$$

whence $X_{fg} = fX_g + gX_f$. Using Cartan's magic formula, we compute:

$$\begin{aligned}
\pounds_{X_{fg}}\Omega &= \pounds_{fX_g+gX_f}\Omega = d\left(\iota_{fX_g+gX_f}\Omega\right) + \iota_{fX_g+gX_f}d\Omega \\
&= d\left(f\iota_{X_g}\Omega\right) + d\left(g\iota_{X_f}\Omega\right) + f\iota_{X_g}d\Omega + g\iota_{X_f}d\Omega \\
&= df\wedge\left(\iota_{X_g}\Omega\right) + fd\left(\iota_{X_g}\Omega\right) + dg\wedge\left(\iota_{X_f}\Omega\right) + gd\left(\iota_{X_f}\Omega\right) + f\iota_{X_g}d\Omega + g\iota_{X_f}d\Omega \\
&= f\left(d\left(\iota_{X_g}\Omega\right) + \iota_{X_g}d\Omega\right) + g\left(d\left(\iota_{X_f}\Omega\right) + \iota_{X_f}d\Omega\right) + df\wedge\left(\iota_{X_g}\Omega\right) + dg\wedge\left(\iota_{X_f}\Omega\right) \\
&= f\pounds_{X_g}\Omega + g\pounds_{X_f}\Omega + df\wedge\left(\iota_{X_g}\Omega\right) + dg\wedge\left(\iota_{X_f}\Omega\right).
\end{aligned}$$

Now note that, since Ω is of top degree: $df\wedge\Omega = 0$. It follows that, for any $Y \in \mathfrak{X}(M)$, we have

$$0 = \iota_Y(df\wedge\Omega) = df(Y)\Omega - df\wedge(\iota_Y\Omega) = Y(f)\Omega - df\wedge(\iota_Y\Omega).$$

In particular,

$$\begin{cases} df\wedge\left(\iota_{X_g}\Omega\right) = X_g(f)\Omega = \{g,f\}\Omega \\ dg\wedge\left(\iota_{X_f}\Omega\right) = X_f(g)\Omega = \{f,g\}\Omega = -\{g,f\}\Omega \end{cases}.$$

So $df\wedge\left(\iota_{X_g}\Omega\right) = -dg\wedge\left(\iota_{X_f}\Omega\right)$ and this implies that $\pounds_{X_{fg}}\Omega = f\pounds_{X_g}\Omega + g\pounds_{X_f}\Omega$. ☐

Example 2.9.4. Let (M^{2n}, Π) be a symplectic manifold, with symplectic form $\omega = -\Pi^{-1}$. Let us compute the modular vector field $X_\Pi^{\omega^n}$ associated with the volume form ω^n. For $f \in C^\infty(M)$, we have

$$X_f = \Pi^\sharp(df) = -(\omega^\flat)^{-1}(df),$$

which implies that $\iota_{X_f}\omega = -df$. Using Cartan's magic formula, we get

$$\pounds_{X_f}\omega = d(\iota_{X_f}\omega) + \iota_{X_f}d\omega = -d^2f + \iota_{X_f}d\omega = 0,$$

where the last equality holds since $d^2 = 0$ and ω is closed. Using the derivation property of $\pounds_{X_f}$, we obtain by induction that also $\pounds_{X_f}\omega^n = 0$. We conclude that $X_\Pi^{\omega^n} = 0$.

Proposition 2.9.5. *Let (M, Π) be an orientable Poisson manifold with volume form Ω. The modular vector field X_Π^Ω is a Poisson vector field.*

Proof. Let $f, g \in C^\infty(M)$. We will prove that

$$X_\Pi^\Omega(\{f,g\}) = \{X_\Pi^\Omega(f), g\} + \{f, X_\Pi^\Omega(g)\}. \tag{2.8}$$

We have

$$\begin{aligned}
X_\Pi^\Omega(\{f,g\})\,\Omega &= \pounds_{X_{\{f,g\}}}\Omega \\
&= \pounds_{[X_f,X_g]}\Omega && \text{(Lemma 2.7.3)} \\
&= \left[\pounds_{X_f}, \pounds_{X_g}\right]\Omega && \text{(Cartan calculus)} \\
&= \pounds_{X_f}\left(\pounds_{X_g}\Omega\right) - \pounds_{X_g}\left(\pounds_{X_f}\Omega\right) \\
&= \pounds_{X_f}\left(X_\Pi^\Omega(g)\Omega\right) - \pounds_{X_g}\left(X_\Pi^\Omega(f)\Omega\right) \\
&= \pounds_{X_f}\left(X_\Pi^\Omega(g)\right)\Omega + X_\Pi^\Omega(g)\pounds_{X_f}\Omega - \pounds_{X_g}\left(X_\Pi^\Omega(f)\right)\Omega - X_\Pi^\Omega(f)\pounds_{X_g}\Omega.
\end{aligned}$$

Note that

$$\begin{cases} X_\Pi^\Omega(f)\pounds_{X_g}\Omega = X_\Pi^\Omega(f)X_\Pi^\Omega(g)\Omega \\ X_\Pi^\Omega(g)\pounds_{X_f}\Omega = X_\Pi^\Omega(g)X_\Pi^\Omega(f)\Omega \end{cases},$$

hence $X_\Pi^\Omega(f)\pounds_{X_g}\Omega = X_\Pi^\Omega(g)\pounds_{X_f}\Omega$. We obtain

$$\begin{aligned}
X_\Pi^\Omega(\{f,g\})\,\Omega &= \pounds_{X_f}\left(X_\Pi^\Omega(g)\right)\Omega - \pounds_{X_g}\left(X_\Pi^\Omega(f)\right)\Omega \\
&= X_f\left(X_\Pi^\Omega(g)\right)\Omega - X_g\left(X_\Pi^\Omega(f)\right)\Omega \\
&= \{f, X_\Pi^\Omega(g)\}\Omega - \{g, X_\Pi^\Omega(f)\}\Omega \\
&= \left[\{X_\Pi^\Omega(f), g\} + \{f, X_\Pi^\Omega(g)\}\right]\Omega,
\end{aligned}$$

which implies what we set out to prove (2.8). □

We now describe how the modular vector field depends on the choice of volume form.

Proposition 2.9.6. *Let (M^n, Π) be an orientable Poisson manifold. Let Ω be a volume form on M with associated modular vector field X_Π^Ω. Changing the volume form Ω changes the modular vector field X_Π^Ω by a hamiltonian vector field:*

$$X_\Pi^{h\Omega} = X_\Pi^\Omega - X_{\log|h|},$$

where $h \in C^\infty(M)$ is a non-vanishing function.

Proof. Since $\wedge^n T^*M$ is a line bundle, any volume form on M is of the form $h\Omega$ for some non-vanishing function $h \in C^\infty(M)$. For any $f \in C^\infty(M)$, we have

$$X_\Pi^{h\Omega}(f) = \frac{\pounds_{X_f}(h\Omega)}{h\Omega} = \frac{X_f(h)\Omega + h\pounds_{X_f}\Omega}{h\Omega} = \frac{X_f(h)}{h} + \frac{\pounds_{X_f}\Omega}{\Omega} = \frac{X_f(h)}{h} + X_\Pi^\Omega(f).$$

Since by the chain rule

$$X_f(\log|h|) = d(\log|h|)(X_f) = \frac{1}{h}dh(X_f) = \frac{1}{h}X_f(h),$$

we get

$$X_\Pi^{h\Omega}(f) = X_f(\log|h|) + X_\Pi^\Omega(f) = -X_{\log|h|}(f) + X_\Pi^\Omega(f).$$

This shows that

$$X_\Pi^{h\Omega} = X_\Pi^\Omega - X_{\log|h|}.$$

□

By Proposition 2.9.5, the modular vector field X_Π^Ω defines a cohomology class $\left[X_\Pi^\Omega\right]$ in the first Poisson cohomology group $H_\Pi^1(M)$. Proposition 2.9.6 implies that this cohomology class does not depend on the chosen volume form: $\left[X_\Pi^\Omega\right] = [X_\Pi] \in H_\Pi^1(M)$.

Definition 2.9.7. The cohomology class $[X_\Pi] \in H_\Pi^1(M)$ of the modular vector field (with respect to any volume form) is called the modular class of M. If this cohomology class is zero, then M is called unimodular.

Example 2.9.8. Example 2.9.4 shows that symplectic manifolds are unimodular.

Remark 2.9.9. On non-orientable Poisson manifolds, one can still define modular vector fields using densities instead of volume forms. We will leave this fact out of consideration.

2.10 Poisson submanifolds

Definition 2.10.1. A Poisson submanifold of a Poisson manifold (M, Π_M) is a Poisson manifold (N, Π_N) together with an injective immersion $i : N \hookrightarrow M$ that is a Poisson map.

One often identifies an immersed submanifold $i : N \hookrightarrow M$ with its image in M, so that one can assume that i is the inclusion map. The tangent space T_xN for $x \in N$ can then be considered as a subspace of T_xM.

Proposition 2.10.2. *Let (M, Π_M) be a Poisson manifold. Given an immersed submanifold $N \hookrightarrow M$, there is at most one Poisson structure Π_N on N that makes (N, Π_N) into a Poisson submanifold. Such Π_N exists, if and only if any of the following equivalent conditions hold:*

1. *$(\Pi_M)_x^\sharp(T_x^*M) \subset T_xN$ for all $x \in N$.*
2. *For every $f \in C^\infty(M)$, the hamiltonian vector field $X_f \in \mathfrak{X}(M)$ is tangent to N.*
3. *For all $x \in N$, the bivector $(\Pi_M)_x$ is tangent to N. That is, $(\Pi_M)_x \in \wedge^2 T_xN$.*

Proof. We first show that the conditions mentioned are equivalent. Since for $f \in C^\infty(M)$ and $x \in N$, we have $(X_f)_x = (\Pi_M)_x^\sharp(d_xf)$, it is clear that 1. and 2. are equivalent.
Next, denoting $d = \dim(M)$ and $s = \dim(N)$, let $(x_1, \ldots, x_d)$ be adapted coordinates around $x \in N$, so that N is locally given by $x_{s+1} = \cdots = x_d = 0$. In these coordinates, we write

$$X_f = \sum_{j=1}^{s}\{f, x_j\}\frac{\partial}{\partial x_j} + \sum_{j=s+1}^{d}\{f, x_j\}\frac{\partial}{\partial x_j}$$

$$\Pi_M = \sum_{1\le i<j\le s} \{x_i, x_j\} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=s+1}^{d} \{x_i, x_j\} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

Assuming that all hamiltonian vector fields on M are tangent to N then implies $\{f, x_j\}(x) = 0$ for $j = s+1, \dots, d$, where f is any smooth function on M locally defined around x. In particular, $\{x_i, x_j\}(x) = 0$ for $i = 1, \dots, d$ and $j = s + 1, \dots, d$. This implies that

$$(\Pi_M)_x = \left(\sum_{1\le i<j\le s} \{x_i, x_j\} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right)_x \in \wedge^2 T_x N.$$

Conversely, if $(\Pi_M)_x \in \wedge^2 T_x N$, then $\{x_i, x_j\}(x) = 0$ for $i = 1, \dots, d$ and $j = s + 1, \dots, d$. Then

$$(X_f)_x = \left(\sum_{i,j=1}^{n} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} \right)_x = \left(\sum_{i=1}^{n} \sum_{j=1}^{s} \{x_i, x_j\} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} \right)_x \in T_x N.$$

This shows that 2. and 3. are equivalent.
Now assume that $i : (N, \Pi_N) \hookrightarrow (M, \Pi_M)$ is a Poisson submanifold. Lemma 2.6.5 gives a commutative diagram

$$\begin{array}{ccc} T_x^* N & \xrightarrow{(\Pi_N^\sharp)_x} & T_x N \\ \uparrow{\scriptstyle (d_x i)^*} & & \downarrow{\scriptstyle d_x i} \\ T_x^* M & \xrightarrow{(\Pi_M^\sharp)_x} & T_x M \end{array} \tag{2.9}$$

That is,

$$d_x i \circ (\Pi_N^\sharp)_x \circ (d_x i)^* = (\Pi_M^\sharp)_x \quad \text{for all } x \in N. \tag{2.10}$$

This shows that $(\Pi_M)_x^\sharp (T_x^* M) \subset d_x i(T_x N)$ for $x \in N$. Since we identify $d_x i(T_x N)$ with $T_x N$, we obtain 1. Also, since $d_x i$ is injective and $(d_x i)^*$ is surjective, the relation (2.10) uniquely determines $(\Pi_N^\sharp)_x$. This proves the uniqueness statement of Proposition 2.10.2.
Conversely, given an immersion $i : N \hookrightarrow M$, assume that $(\Pi_M)_x^\sharp (T_x^* M) \subset T_x N$ for all $x \in N$. The natural vector bundle map $i^* : T^* M|_N \to T^* N$ is surjective, whence it has a splitting $j : T^* N \to T^* M|_N$. We define $\Pi_N^\sharp := \Pi_M^\sharp|_N \circ j$, which is a map $T^* N \to TN$ by the assumption. It is smooth by composition, and its definition does not depend on the choice of splitting. Indeed, let $x \in N$ and $\alpha, \beta, \theta \in T_x^* M$ such that $\alpha|_{T_x N} = \beta|_{T_x N}$. Then

$$\big\langle (\Pi_M^\sharp)_x(\alpha) - (\Pi_M^\sharp)_x(\beta), \theta \big\rangle = \big\langle (\Pi_M^\sharp)_x(\alpha - \beta), \theta \big\rangle = -\big\langle (\Pi_M^\sharp)_x(\theta), \alpha - \beta \big\rangle = 0,$$

as by assumption $(\Pi_M^\sharp)_x(\theta) \in T_x N$ and $(\alpha - \beta)|_{T_x N} = 0$. By construction, $\Pi_N^\sharp$ makes the diagram (2.9) commute. It remains to show that Π_N is Poisson, i.e. that $[\Pi_N, \Pi_N] = 0$.
Commutativity of the diagram (2.9) implies that Π_N and Π_M are i-related. (Use Lemmas 2.6.5 and 2.6.3. Note that at this point, Π_N is only almost Poisson, whereas the aforementioned Lemmas are stated for Poisson structures. Notice however that their statements nor their proofs make use of the Jacobi identity or of the Schouten bracket being zero, which indicates that they hold more generally for almost Poisson structures.) By Lemma 8.3.2 in the appendix, also $[\Pi_N, \Pi_N]$ and $[\Pi_M, \Pi_M]$ are i-related. As $[\Pi_M, \Pi_M] = 0$ this means that

$$(d_x i)[\Pi_N, \Pi_N]_x = 0 \text{ for all } x \in N.$$

This in turn implies that $[\Pi_N,\Pi_N]_x = 0$ at $x \in N$. Indeed, surjectivity of $(d_x i)^* : T_x^*M \to T_x^*N$ gives that for any $f_1, f_2, f_3 \in C^\infty(N)$:

$$\begin{aligned}
[\Pi_N,\Pi_N]_x(d_xf_1, d_xf_2, d_xf_3) &= [\Pi_N,\Pi_N]_x\left((d_xi)^*\alpha_1, (d_xi)^*\alpha_2, (d_xi)^*\alpha_3\right) \\
&= [\Pi_N,\Pi_N]_x\left(\alpha_1\circ d_xi, \alpha_2\circ d_xi, \alpha_3\circ d_xi\right) \\
&= \big((d_xi)[\Pi_N,\Pi_N]_x\big)(\alpha_1,\alpha_2,\alpha_3) \\
&= 0.
\end{aligned}$$

We conclude that $[\Pi_N,\Pi_N] = 0$, hence Π_N is the unique Poisson structure on N that makes N into a Poisson submanifold of M. □

Example 2.10.3. If (M,Π) is symplectic, then $\Pi_x^\sharp(T_x^*M) = T_xM$ for all $x \in M$. It follows that the only Poisson submanifolds of (M,Π) are open subsets of M. By contrast, there are many more symplectic submanifolds of M (for instance, any point $\{p\} \subset M$).

2.11 The splitting theorem

The splitting theorem is a normal form theorem that describes Poisson structures locally. It generalizes the Darboux theorem for symplectic manifolds to arbitrary Poisson manifolds. We will need the following lemma.

Lemma 2.11.1. *If M is an m-dimensional manifold and $X_1,\ldots,X_n$ are vector fields defined on an open subset $U \subset M$ such that:*

1. *$\{X_1(q),\ldots,X_n(q)\}$ is linearly independent at each $q \in U$;*
2. *$[X_i,X_j] = 0$ on U, for $1 \le i,j \le n$,*

then any $p \in U$ has a coordinate neighborhood $(V, x_1,\ldots,x_m)$ such that $X_j = \frac{\partial}{\partial x_j}$ on V, for $j = 1,\ldots,n$.

Proof. See [Sh]. □

Theorem 2.11.2 (Weinstein's splitting theorem)**.** *Let (M,Π) be a Poisson manifold, and $x_0 \in M$ with rank$(\Pi_{x_0}) = 2k$. Then there exists a coordinate system $(p_1,\ldots,p_k,q_1,\ldots,q_k,y_1,\ldots,y_l)$ centered at x_0 such that*

$$\Pi = \sum_{i=1}^k \frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_i} + \sum_{1\le i<j\le l}\phi_{i,j}(y_1,\ldots,y_l)\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j},$$

and $\phi_{i,j}(0,\ldots,0) = 0$.

Proof. If rank$(\Pi_{x_0}) = 0$, then any coordinate system $(y_1,\ldots,y_n)$ centered at x_0 satisfies the theorem. So assume that rank$(\Pi_{x_0}) > 0$, i.e. $\Pi^\sharp_{x_0} : T^*_{x_0}M \to T_{x_0}M$ is not the zero map. This implies that there exists a function f, locally defined around x_0, such that $\Pi^\sharp(df) = X_f$ is non-vanishing. Lemma 2.11.1 implies the existence of coordinates $(p_1,\ldots)$ around x_0 such that $X_f = \frac{\partial}{\partial p_1}$. We then have

$$\{f,p_1\} = X_f(p_1) = \frac{\partial}{\partial p_1}p_1 = 1, \tag{2.11}$$

hence

$$[X_f, X_{p_1}] = X_{\{f,p_1\}} = X_1 = 0.$$

Here, the first equality is Lemma 2.7.3 and the last equality follows from the Leibniz identity for the bracket $\{\cdot,\cdot\}$. Define $q_1 := f$.
Note that X_{q_1} and X_{p_1} are linearly independent everywhere, since $X_{q_1}(x) = \lambda X_{p_1}(x)$ would imply

$$\{q_1, p_1\}(x) = X_{q_1}(x)(p_1) = \lambda X_{p_1}(x)(p_1) = \lambda\{p_1, p_1\}(x) = 0,$$

which contradicts (2.11). We can again use Lemma 2.11.1 to find coordinates $(y_1, \dots, y_n)$ around x_0 such that $X_{p_1} = \frac{\partial}{\partial y_1}$ and $X_{q_1} = \frac{\partial}{\partial y_2}$. We now take $(p_1, q_1, y_3, \dots, y_n)$ as a new system of coordinates. Indeed, the map $\psi : (y_1, \dots, y_n) \mapsto (p_1, q_1, y_3, \dots, y_n)$ has Jacobian matrix

$$\begin{bmatrix} \frac{\partial p_1}{\partial y_1} & \frac{\partial p_1}{\partial y_2} & \cdots\cdots\cdots\cdots & \\ \frac{\partial q_1}{\partial y_1} & \frac{\partial q_1}{\partial y_2} & \cdots\cdots\cdots\cdots & \\ \frac{\partial y_3}{\partial y_1} & \frac{\partial y_3}{\partial y_2} & \frac{\partial y_3}{\partial y_3} & \cdots & \frac{\partial y_3}{\partial y_n} \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{\partial y_n}{\partial y_1} & \frac{\partial y_n}{\partial y_2} & \frac{\partial y_n}{\partial y_3} & \cdots & \frac{\partial y_n}{\partial y_n} \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & \bigstar \\ -1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & I_{(n-2)\times(n-2)} \end{array}\right],$$

which follows from the computations

$$\begin{cases} \frac{\partial p_1}{\partial y_1} = X_{p_1}(p_1) = \{p_1, p_1\} = 0 \\ \frac{\partial p_1}{\partial y_2} = X_{q_1}(p_1) = \{q_1, p_1\} = 1 \\ \frac{\partial q_1}{\partial y_1} = X_{p_1}(q_1) = \{p_1, q_1\} = -1 \\ \frac{\partial q_1}{\partial y_2} = X_{q_1}(q_1) = \{q_1, q_1\} = 0. \end{cases}$$

The latter matrix has nonzero determinant (equal to 1), which shows that ψ is indeed a change of coordinates. In these coordinates, we have

- $\{q_1, p_1\} = 1$;
- $\{p_1, y_i\} = X_{p_1}(y_i) = \frac{\partial y_i}{\partial y_1} = 0$ for $i \geq 3$;
- $\{q_1, y_i\} = X_{q_1}(y_i) = \frac{\partial y_i}{\partial y_2} = 0$ for $i \geq 3$.

Hence

$$\Pi = \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \sum_{3 \leq i < j \leq n} \phi_{i,j}(p_1, q_1, y_3, \dots, y_n) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j},$$

where $\phi_{i,j} = \{y_i, y_j\}$. In fact, the $\phi_{i,j}$ don't depend on the variables p_1, q_1:

$$\begin{aligned} \frac{\partial}{\partial p_1}\phi_{i,j} &= \frac{\partial}{\partial p_1}\{y_i, y_j\} = X_{q_1}\{y_i, y_j\} = \{q_1, \{y_i, y_j\}\} \\ &= \{\{y_j, q_1\}, y_i\} + \{\{q_1, y_i\}, y_j\} \qquad \text{(Jacobi identity)} \\ &= 0. \end{aligned}$$

Also, note that $X_{p_1} = -\frac{\partial}{\partial q_1}$ (use Lemma 2.7.4). Hence

$$\begin{aligned} \frac{\partial}{\partial q_1}\phi_{i,j} &= \frac{\partial}{\partial q_1}\{y_i, y_j\} = -X_{p_1}\{y_i, y_j\} = -\{p_1, \{y_i, y_j\}\} \\ &= \{y_i, \{y_j, p_1\}\} + \{y_j, \{p_1, y_i\}\} \qquad \text{(Jacobi identity)} \\ &= 0. \end{aligned}$$

Hence,

$$\Pi = \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \sum_{3 \leq i < j \leq n} \phi_{i,j}(y_3, \dots, y_n) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} := \frac{\partial}{\partial q_1} \wedge \frac{\partial}{\partial p_1} + \Pi'.$$

Now note that Π' is a Poisson structure on an $(n-2)$-dimensional manifold with coordinates $(y_3, \dots, y_n)$. Indeed, let $\{\cdot,\cdot\}$ be the bracket corresponding to Π and $\{\cdot,\cdot\}'$ the bracket corresponding to $\Pi^{'}$ (a priori, the latter is only almost Poisson). If f, g are functions only depending on the coordinates $y_3, \dots, y_n$, then we have

$$\{f, g\} = \Pi(df, dg) = \Pi'(df, dg) = \{f, g\}'.$$

Since $\{\cdot,\cdot\}$ satisfies the Jacobi identity, this implies that also $\{\cdot,\cdot\}'$ satisfies the Jacobi identity. Hence we can repeat our argument for the Poisson structure Π', and conclude by induction on the rank of Π at x_0. □

Remark 2.11.3. The splitting theorem states that around any point x, a Poisson manifold is a direct product of a symplectic manifold (with symplectic form $\sum_{i=1}^{k} dq_i \wedge dp_i$), and a transverse Poisson manifold with Poisson structure vanishing at x. This explains how the theorem received its name: a Poisson structure splits locally into a non-degenerate part and a singular part.

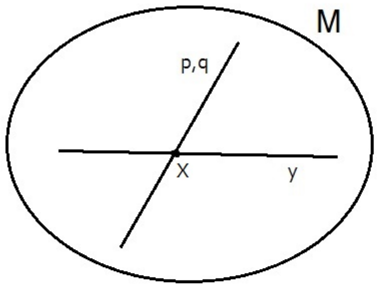


Figure 2.1: The Poisson structure on M locally splits into a symplectic structure on the locus $\{y = 0\}$ and a transverse Poisson structure vanishing at x.

2.12 The symplectic foliation

In this section, we will show that a Poisson manifold is naturally partitioned into symplectic manifolds. The appropriate notion in this context is that of a foliation.

Definition 2.12.1. A singular foliation of a manifold M is a partition $\mathcal{F} = \{\mathcal{F}_\alpha\}$ of M in immersed connected submanifolds $\mathcal{F}_\alpha$, called leaves, that satisfies the following property around any $x \in M$: if $\mathcal{F}_x$ is the leaf containing x, and $m = \dim(M)$ and $d = \dim(\mathcal{F}_x)$, then there exists a chart $h = (y_1, \dots, y_m) : U \to (-\epsilon, \epsilon)^m$ such that the path connected component of $\mathcal{F}_x \cap U$ containing x is given by $\{y_{d+1} = \dots = y_m = 0\}$, and each level set $\{y_{d+1} = c_{d+1}, \dots, y_m = c_m\}$ (where $c_{d+1}, \dots, c_m$ are constants) is completely contained in some leaf $\mathcal{F}_\alpha$ of $\mathcal{F}$. If all leaves $\mathcal{F}_\alpha$ of $\mathcal{F}$ have the same dimension, then $\mathcal{F}$ is called a regular foliation.

Remark 2.12.2. Phrased differently, a regularly foliated manifold M is locally modelled as an affine space decomposed into parallel affine subspaces.

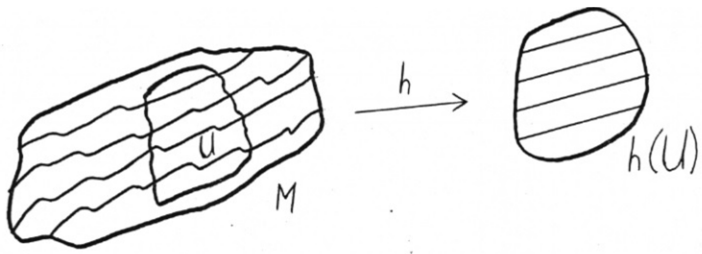


Figure 2.2: A regular 1-dimensional foliation. Figure taken from [Mil]

Definition 2.12.3. A singular distribution $\Delta \subset TM$ is the assignment to each $x \in M$ of a subspace $\Delta_x \subset T_xM$, i.e.

$$\Delta = \bigsqcup_{x \in M} \Delta_x.$$

A singular distribution Δ is smooth if for all $x \in M$ and $v \in \Delta_x$, there exists a vector field X locally defined around x, such that X is tangent to Δ and $X(x) = v$. If $\dim(\Delta_x)$ is independent of x, then the distribution Δ is called regular.

Example 2.12.4. Let $\mathcal{F}$ be a singular foliation. If $\mathcal{F}_x$ denotes the leaf of $\mathcal{F}$ containing x, then let $\Delta_x^{\mathcal{F}} := T_x\mathcal{F}_x$. This defines a smooth singular distribution $\Delta^{\mathcal{F}}$, called the tangent distribution of the foliation $\mathcal{F}$.

Definition 2.12.5. An integral submanifold of a smooth singular distribution Δ on M is a connected immersed submanifold $N \hookrightarrow M$ such that $T_qN = \Delta_q$ for all $q \in N$. An integral submanifold is maximal if it is not contained in any strictly larger integral submanifold.

If a point $x \in M$ is contained in an integral submanifold of Δ, then it is contained in a unique maximal one.

Definition 2.12.6. A smooth singular distribution Δ on M is integrable if every point of M is contained in an integral submanifold of Δ. If this is the case, then each point lies in a unique maximal integral submanifold, so the maximal integral submanifolds form a partition of M.

Definition 2.12.7. A smooth singular distribution Δ on M is generated by a family $\mathcal{C}$ of vector fields if at each point $x \in M$, Δ_x is spanned by the values at x of the vector fields of $\mathcal{C}$.
A distribution Δ is invariant with respect to a family $\mathcal{C}$ of vector fields if for all $X \in \mathcal{C}$:

$$(d_x\phi_t)\Delta_x = \Delta_{\phi_t(x)}. \tag{2.12}$$

Here (ϕ_t) is the local flow of X, and Equation (2.12) has to hold wherever $\phi_t(x)$ is defined.

The classical Stefan-Sussmann theorem relates these concepts.

Theorem 2.12.8 (Stefan-Sussmann)**.** *Let Δ be a smooth singular distribution on M. The following are equivalent:*

1. *Δ is integrable.*
2. *Δ is generated by a family $\mathcal{C}$ of vector fields, and is invariant with respect to $\mathcal{C}$.*
3. *Δ is the tangent distribution $\Delta^{\mathcal{F}}$ of a singular foliation $\mathcal{F}$.*

Proof. See for instance [DT]. □

Definition 2.12.9. A distribution Δ is involutive if it is closed under the Lie bracket. That is, if X, Y are vector fields tangent to Δ, then also their Lie bracket $[X, Y]$ is tangent to Δ.

Example 2.12.10. If $X \in \mathfrak{X}(M)$ is a non-vanishing vector field, then $\text{span}\{X\}$ is a smooth regular involutive distribution by skew-symmetry of the Lie bracket.

We now focus on regular distributions.

Definition 2.12.11. Let Δ be a regular distribution of dimension n on M^{n+k}. We say that Δ is completely integrable if each point $p \in M$ has a coordinate neighborhood $(U, x_1, \ldots, x_{n+k})$ such that $\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}$ is a local basis for Δ on U.

One clearly has

$$\text{completely integrable} \Rightarrow \text{integrable} \Rightarrow \text{involutive}.$$

Indeed, if $q \in M$ is contained in a coordinate neighborhood as in Definition 2.12.11, and q has coordinates $(a_1, \ldots, a_{n+k})$, then the slice $x_{n+1} = a_{n+1}, \ldots, x_{n+k} = a_{n+k}$ is an integral submanifold through q. As for the second implication, we take $X, Y \in \Gamma(\Delta)$ defined near q. Let N be an integral submanifold of Δ containing q. Then X and Y are tangent to N, hence so is their Lie bracket $[X, Y]$ (see Lemma 4.1.9). Therefore, $[X, Y](q) \in \Delta_q$.

The classical Frobenius theorem states that the implications above are actually equivalences:

$$\text{completely integrable} \Leftrightarrow \text{integrable} \Leftrightarrow \text{involutive}.$$

Theorem 2.12.12 (Frobenius). *A smooth regular distribution is involutive if and only if it is completely integrable.*

Proof. See for instance [Lee]. □

Remark 2.12.13. Frobenius' theorem does not hold for singular distributions. For instance, consider $M = \mathbb{R}^2$ with distribution

$$\Delta_{(x,y)} = \begin{cases} T_{(x,y)}\mathbb{R}^2 & \text{if } x > 0 \\ \text{span}\{\frac{\partial}{\partial x}\} & \text{if } x \leq 0 \end{cases} .$$

Let $X = \frac{\partial}{\partial x}$ and $Y = f\frac{\partial}{\partial y}$, where f is defined by

$$f(x, y) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x^2} & \text{if } x > 0 \end{cases} .$$

Then Δ is generated by X and Y, so Δ is smooth. Note also that Δ is involutive since

$$\left[\frac{\partial}{\partial x}, f\frac{\partial}{\partial y}\right] = \frac{\partial f}{\partial x}\frac{\partial}{\partial y} = \begin{cases} 0 & \text{if } x \leq 0 \\ \frac{2}{x^3}e^{-1/x^2}\frac{\partial}{\partial y} = \frac{2}{x^3}f\frac{\partial}{\partial y} & \text{if } x > 0 \end{cases} .$$

But Δ is not integrable since we cannot find leaves through points on the y-axis.

We now specialize to Poisson manifolds.

Definition 2.12.14. Let (M, Π) be a Poisson manifold. It has a characteristic distribution Δ defined by

$$\Delta_x = \Pi_x^\sharp(T_x^*M) = \{X_f(x) : f \in C^\infty(M)\}.$$

It is clear that Δ is a smooth distribution, generated by the set of hamiltonian vector fields.

Lemma 2.12.15. *The characteristic distribution* Δ *of a Poisson manifold* (M, Π) *is integrable.*

Proof. By the Stefan-Sussmann theorem, it is enough to show that Δ is invariant with respect to the family of hamiltonian vector fields. Choose $f \in C^\infty(M)$ and let (ϕ_t) be the local flow of the hamiltonian vector field X_f. By Remark 2.7.7, we know that X_f is a Poisson vector field, and thus by Definition 2.7.5 we get that (ϕ_t) consists of Poisson diffeomorphisms. Lemma 2.6.5 gives that

$$(d_x\phi_t) \circ \Pi_x^\sharp \circ (d_x\phi_t)^* = \Pi^\sharp_{\phi_t(x)}. \tag{2.13}$$

Note here that $d_x\phi_t$ and $(d_x\phi_t)^*$ are isomorphisms since ϕ_t is a diffeomorphism. In particular, $(d_x\phi_t)^*$ is surjective, whence Equation (2.13) implies that

$$(d_x\phi_t)\Delta_x = \Delta_{\phi_t(x)}.$$

□

Hence, we find a singular foliation $\mathcal{F}$ that integrates the characteristic distribution Δ. If $\mathcal{F}_x$ is the leaf containing $x \in M$, then

$$\Pi_x^\sharp(T_x^*M) = \Delta_x = T_x\mathcal{F}_x. \tag{2.14}$$

Hence, by Proposition 2.10.2 we get that $\mathcal{F}_x$ is a Poisson submanifold of (M, Π), with uniquely determined Poisson structure $\Pi_{\mathcal{F}_x}$ that is the restriction to $\mathcal{F}_x$ of the original Poisson structure Π. That is,

$$\{f, g\}(y) = \{f|_{\mathcal{F}_x}, g|_{\mathcal{F}_x}\}_{\mathcal{F}_x}(y) \quad \text{for } f, g \in C^\infty(M) \text{ and } y \in \mathcal{F}_x.$$

Indeed, the inclusion $i : \mathcal{F}_x \hookrightarrow M$ is a Poisson map, and thus

$$\{f, g\} \circ i = \{f \circ i, g \circ i\}_{\mathcal{F}_x}.$$

Finally, since the inclusion $i : \mathcal{F}_x \hookrightarrow M$ is Poisson, Lemma 2.6.5 implies that

$$\Pi_y^\sharp(\alpha) = (\Pi_{\mathcal{F}_x})_y^\sharp(\alpha \circ d_y i)$$

for $y \in \mathcal{F}_x$ and $\alpha \in T_y^*M$. Together with Equation (2.14), this gives that $(\Pi_{\mathcal{F}_x})_y^\sharp(T_y^*\mathcal{F}_x) = T_y\mathcal{F}_x$. That is, $\Pi_{\mathcal{F}_x}$ is of maximal rank and thus defines a symplectic structure on the leaf $\mathcal{F}_x$.

We summarize:

Proposition 2.12.16. *A Poisson manifold* (M, Π) *is foliated into Poisson submanifolds, whose tangent spaces are spanned by the hamiltonian vector fields of* (M, Π). *The restriction of the Poisson structure* Π *to each of these submanifolds is symplectic. We call this decomposition* $\mathcal{F}$ *the symplectic foliation of* M*, and the immersed submanifolds are the symplectic leaves of* M.

The Poisson structure Π is completely determined by the symplectic structures on the leaves of $\mathcal{F}$.

Proposition 2.12.17. *Let* Π_1 *and* Π_2 *be two Poisson structures on a manifold* M. *Suppose that both Poisson structures define the same foliation on* M *and that for every leaf* $\mathcal{L}$*, the Poisson (symplectic) structure induced on* $\mathcal{L}$ *by* Π_1 *is the same as the Poisson (symplectic) structure induced on* $\mathcal{L}$ *by* Π_2. *Then* Π_1 *and* Π_2 *are equal.*

Proof. Let $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ be the Poisson brackets corresponding to Π_1 and Π_2 respectively. By assumption, they determine the same symplectic foliation on M. Let $p \in M$ and let $\mathcal{L}$ be the leaf passing through p. Denote by $i : \mathcal{L} \hookrightarrow M$ the inclusion map. It is assumed that the induced Poisson structures $\{\cdot, \cdot\}_{1,\mathcal{L}}$ and $\{\cdot, \cdot\}_{2,\mathcal{L}}$ on $\mathcal{L}$ coincide, so that for all $f, g \in C^\infty(M)$:

$$\{f \circ i, g \circ i\}_{1,\mathcal{L}} = \{f \circ i, g \circ i\}_{2,\mathcal{L}}.$$

Since i is a Poisson map, both when M is equipped with Π_1 and when M is equipped with Π_2, we have

$$\{f,g\}_1(p) = \{f,g\}_1(i(p)) = \{f\circ i, g\circ i\}_{1,\mathcal{L}}(p) = \{f\circ i, g\circ i\}_{2,\mathcal{L}}(p) = \{f,g\}_2(i(p)) = \{f,g\}_2(p).$$

This applies to any $p \in M$ and all $f, g \in C^\infty(M)$, so that $\{\cdot,\cdot\}_1 = \{\cdot,\cdot\}_2$. □

Remark 2.12.18. In case Π is a regular Poisson structure on M, we can readily argue for the symplectic foliation of M as follows. Invoking Lemma 2.7.3, we see that the characteristic distribution Δ of (M,Π) is involutive. Using the Frobenius theorem, we find the desired foliation $\mathcal{F}$ integrating Δ.

Example 2.12.19. Consider $\mathbb{R}^2$ with Poisson structure $\Pi = y\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial x}$. Since

$$\Pi^\sharp(dx) = -y\frac{\partial}{\partial y} \quad \text{and} \quad \Pi^\sharp(dy) = y\frac{\partial}{\partial x},$$

we obtain that the characteristic distribution Δ is given by

$$\Delta_{(x,y)} = \begin{cases} T_{(x,y)}\mathbb{R}^2 & \text{if } y \neq 0 \\ \{0\} & \text{if } y = 0 \end{cases}.$$

Hence, the symplectic leaves of $(\mathbb{R}^2,\Pi)$ are the open upper half plane, the open lower half plane and all points on the x-axis.

Chapter 3

Basic features of log-symplectic structures

We will now introduce log-symplectic structures, which are the objects of study in this thesis. Log-symplectic manifolds form a convenient class of Poisson manifolds that extends the class of symplectic manifolds. Log-symplectic structures are however in many ways equally well-behaved as honest symplectic structures, and many results from symplectic geometry can be extended to the log-symplectic framework. This made log-symplectic structures a topic of intense research in the Poisson community in the last couple of years.

This chapter is a compilation of basic results and examples taken from various sources. However, it also contains an important normal form result: we give a particularly neat coordinate expression for a log-symplectic structure near a point of its singular locus.

3.1 Definition and examples

Recall that a symplectic structure on a manifold M^{2n} corresponds to a non-degenerate Poisson structure on M^{2n}, that is, a Poisson bivector Π whose top wedge power Π^n is nowhere vanishing. We will now relax this condition by allowing Π^n to vanish linearly. This leads to the following definition.

Definition 3.1.1. A Poisson structure Π on a manifold M^{2n} is called log-symplectic if the map

$$\Pi^n : M \to \bigwedge^{2n} TM : x \mapsto \Pi^n(x)$$

is transverse to the zero section of $\bigwedge^{2\mathrm{n}} TM$. We call $Z = (\Pi^n)^{-1}(0)$ its singular locus.

Remark 3.1.2. Honest symplectic structures are also log-symplectic. Our interest goes out to the *bona fide* or non-symplectic log-symplectic structures.

The transversality condition implies that the singular locus Z is a codimension-one submanfold of M. Indeed, we have that $\Pi^n(M)$ and M are $2n$-dimensional submanifolds of $\bigwedge^{2\mathrm{n}} TM$, and it is a well-known fact in differential geometry that their transverse intersection $\Pi^n(M) \cap M$ is a smooth submanifold of M, of dimension

$$\dim\left(\Pi^n(M)\right) + \dim(M) - \dim\left(\bigwedge^{2n} TM\right) = 2n + 2n - (2n+1) = 2n - 1.$$

Here we used that $\bigwedge^{2\mathrm{n}} TM$ is a vector bundle of rank 1 over M, whence its dimension is $2n+1$.

It is now apparent that log-symplectic structures (M, Z, Π) are not far from being symplectic, as they are symplectic on the open dense subset $M \setminus Z$ of M. Moreover, their failure of being symplectic everywhere is as well-behaved as one can ask, since the zeros of Π^n are simple and forced to lie in the hypersurface Z.

Remark 3.1.3. Given a Poisson manifold (M^{2n}, Π), we check if Π is log-symplectic as follows: Choose $p \in (\Pi^n)^{-1}(0)$. Let $(U, x_1, \ldots, x_{2n})$ be coordinates around p, so that on U

$$\Pi^n = g\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_{2n}},$$

for some smooth function $g \in C^\infty(U)$ vanishing at p. Through the choice of coordinates, we obtain a local trivialization of $\bigwedge^{2\mathrm{n}} TM$ around p, on which the map Π^n is given by

$$\Pi^n : U \to U \times \mathbb{R} : x \mapsto (x, g(x)).$$

Then Π^n intersects the zero section transversely at p

$$\begin{aligned}
&\Leftrightarrow \mathrm{Im}\,(d_p\Pi^n) \oplus T_{(p,0)}\big(U \times \{0\}\big) = T_{(p,0)}\big(U \times \mathbb{R}\big)\\
&\Leftrightarrow \big(\mathbb{R}^{2n} \times \mathrm{Im}\,(d_p g)\big) \oplus \big(\mathbb{R}^{2n} \times \{0\}\big) = \mathbb{R}^{2n} \times \mathbb{R}\\
&\Leftrightarrow \mathrm{Im}\,(d_p g) = \mathbb{R}\\
&\Leftrightarrow d_p g \neq 0\\
&\Leftrightarrow \text{g vanishes linearly at } p.
\end{aligned}$$

Now assume that M is orientable. Let Ω be a volume form on M with dual $2n$-vector field ξ. Since $\bigwedge^{2\mathrm{n}} TM$ is a line bundle, we can write

$$\Pi^n = f\xi,$$

for uniquely determined $f \in C^\infty(M)$. Let p be a zero of Π^n. As before, choosing coordinates $(U, x_1, \ldots, x_{2n})$ around p, we write

$$\Pi^n = g\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_{2n}}.$$

As

$$\frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_{2n}} = h\xi|_U$$

for some non-vanishing function $h \in C^\infty(U)$, we get on U that

$$\Pi^n = gh\xi|_U = f|_U\xi|_U.$$

Since g vanishes at p, the Leibniz rule gives

$$d_p f = h(p)d_p g + g(p)d_p h = h(p)d_p g,$$

where $h(p)$ is nonzero. Keeping in mind the above discussion, we thus obtain that Π^n intersecting the zero section transversely at p is equivalent with $d_p f$ being nonzero. Noting that

$$Z = (\Pi^n)^{-1}(0) = f^{-1}(0),$$

we conclude:

Corollary 3.1.4. *Let (M^{2n},Π) be an orientable Poisson manifold and assume that $\Pi^n = f\xi$, where ξ is the dual $2n$-vector field of some volume form Ω. Then (M^{2n},Π) is log-symplectic if and only if 0 is a regular value of f.*

In case M is log-symplectic and orientable, this gives an alternative proof of the fact that its singular locus Z is a codimension-one submanifold. Indeed, it is the preimage of a regular value under a map $f \in C^\infty(M)$.

The following example shows that the class of log-symplectic manifolds is strictly larger than the class of symplectic manifolds.

Example 3.1.5 (Following [FM])**.** Consider the unit sphere $S^2 \subset \mathbb{R}^3$, where $\mathbb{R}^3$ is endowed with cylindrical coordinates (r,θ,z). In these coordinates, S^2 is described by $r^2+z^2=1$, and (θ,z) are the induced coordinates on S^2. We now claim that $\omega := d\theta\wedge dz$ is a well-defined, non-degenerate differential form on S^2. Indeed, even though $d\theta$ is ill-defined at the north and south pole, $d\theta\wedge dz$ extends smoothly over the poles. We will prove this for the north pole, using Cartesian coordinates.
On the open upper hemisphere, we have $z=\sqrt{1-x^2-y^2}$ and thus

$$dz = -\frac{x}{\sqrt{1-x^2-y^2}}dx - \frac{y}{\sqrt{1-x^2-y^2}}dy.$$

Using that $\theta = \arctan(y/x)$, we get that

$$d\theta = \frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx.$$

Hence,

$$\begin{aligned} d\theta\wedge dz &= \left(\frac{x}{x^2+y^2}\right)\left(\frac{x}{\sqrt{1-x^2-y^2}}\right)dx\wedge dy + \left(\frac{y}{x^2+y^2}\right)\left(\frac{y}{\sqrt{1-x^2-y^2}}\right)dx\wedge dy \\ &= \frac{(x^2+y^2)}{(x^2+y^2)\sqrt{1-x^2-y^2}}dx\wedge dy. \end{aligned}$$

So the singularity at the north pole is removable, and $d\theta\wedge dz$ extends smoothly as

$$\left.\left(\frac{1}{\sqrt{1-x^2-y^2}}dx\wedge dy\right)\right|_{(0,0)} = dx\wedge dy|_{(0,0)}. \tag{3.1}$$

One proceeds similarly around the south pole, using that $z=-\sqrt{1-x^2-y^2}$. From equation (3.1) and its analog around the south pole, it is clear that $d\theta\wedge dz$ is non-vanishing at the poles. Away from the poles, it is obviously non-vanishing, whence $\omega = d\theta\wedge dz$ is a non-degenerate 2-form on S^2, as we claimed. Consequently, it has an inverse bivector field ω^{-1}. This allows us to define a bivector Π on S^2 by

$$\Pi := -z\omega^{-1} = z\frac{\partial}{\partial\theta}\wedge\frac{\partial}{\partial z}.$$

Being a bivector on a surface, Π is automatically Poisson (see Example 2.4.7). By Remark 3.1.3, it follows that Π is a log-symplectic structure on S^2.
Now consider the antipodal action of $\mathbb{Z}_2=\{0,1\}$ on S^2, that is:

$$1\cdot(\theta,z) = (\theta+\pi,-z).$$

Note that Π is invariant under this action. Indeed, defining $\phi : (\theta, z) \mapsto (\theta + \pi, -z)$, we show that $\phi_* \Pi = \Pi$. Since the Jacobian matrix of ϕ at any point is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

it follows that

$$\phi_* \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta} \quad \text{and} \quad \phi_* \frac{\partial}{\partial z} = -\frac{\partial}{\partial z}.$$

Noting that by definition, $\phi_* z = (\phi^{-1})^* z = -z$, we get

$$\phi_* \left(z \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial z} \right) = (\phi_* z) \left(\phi_* \frac{\partial}{\partial \theta} \right) \wedge \left(\phi_* \frac{\partial}{\partial z} \right) = (-z) \frac{\partial}{\partial \theta} \wedge \left(-\frac{\partial}{\partial z} \right) = z \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial z}.$$

Consequently, Π descends to a Poisson structure $\overline{\Pi}$ on the orbit space $S^2/\mathbb{Z}_2 \cong \mathbb{RP}^2$. As the projection $S^2 \to \mathbb{RP}^2$ is a local diffeomorphism (it is a covering map) and being log-symplectic is a local property, it follows that $\overline{\Pi}$ is a log-symplectic structure on $\mathbb{RP}^2$.
However, $\mathbb{RP}^2$ is not symplectic since it is not orientable. Thus $\mathbb{RP}^2$ is a *bona fide* log-symplectic manifold.

The standard example of a log-symplectic structure is the following.

Example 3.1.6. Consider $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n)$. The bivector Π defined as

$$\Pi = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} \tag{3.2}$$

is a log-symplectic structure on $\mathbb{R}^{2n}$. Let us first check that Π is Poisson (i.e. $[\Pi, \Pi] = 0$), using the defining properties of the Schouten bracket (Theorem 2.2.2). We have

$$\begin{aligned}
[\Pi, \Pi] &= \left[y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}, y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} \right] \\
&= \left[y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1}, y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} \right] + 2 \left[y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1}, \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} \right] \\
&\quad + \left[\sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}, \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} \right].
\end{aligned}$$

The graded derivation property of $[\cdot, \cdot]$ reduces the last term to Lie brackets of coordinate vector fields, which vanish. Application of the derivation property and Lemma 2.2.5 then gives

$$\begin{aligned}
[\Pi, \Pi] &= \left[y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1}, y_1 \right] \wedge \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + y_1 \left[y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1}, \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} \right] \\
&\quad + 2 \left[\sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}, y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} \right] \\
&= -\iota_{dy_1} \left(y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} \right) \wedge \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + y_1 \left[\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1}, y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} \right] \\
&\quad + 2 \left[\sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}, y_1 \right] \wedge \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + 2 y_1 \left[\sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}, \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} \right]
\end{aligned}$$

As before, the last term vanishes. So

$$
\begin{aligned}
[\Pi,\Pi] &= y_1\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial y_1}+y_1\left(\left[\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial y_1},y_1\right]\wedge\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial y_1}+y_1\left[\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial y_1},\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial y_1}\right]\right)\\
&\quad -2\iota_{dy_1}\left(\sum_{i=2}^{n}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial y_i}\right)\wedge\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial y_1}\\
&= -y_1\iota_{dy_1}\left(\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial y_1}\right)\wedge\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial y_1}=y_1\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial y_1}=0.
\end{aligned}
$$

Next, as

$$
\Pi^n = n!\,y_1\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial y_1}\wedge\cdots\wedge\frac{\partial}{\partial x_n}\wedge\frac{\partial}{\partial y_n},
$$

Remark 3.1.3 shows that Π^n is transverse to the zero section. Hence Π is indeed a log-symplectic structure on $\mathbb{R}^{2n}$. Its singular locus is the hyperplane $Z \leftrightarrow \{y_1 = 0\}$.

Example 3.1.6 is prototypical in the sense that every log-symplectic structure looks like that near its singular locus. We will prove this in the next section.

3.2 Normal form

We will now prove that the expression (3.2) in Example 3.1.6 is a local normal form for log-symplectic structures near their singular locus. The following lemma is stated in [GMP2], without proof however.

Lemma 3.2.1. *Let (M^{2n}, Z, Π) be a log-symplectic manifold. The rank of Π at any point $x \in Z$ equals $2n-2$.*

Proof. Choose $x \in Z$. Since Π^n vanishes at x, we have that Π_x is not of full rank $2n$. By skew-symmetry, its rank is even, whence at most $2n-2$. We will assume by contradiction that $\text{rank}(\Pi_x) = 2k < 2n-2$. By Weinstein's splitting theorem, we find coordinates $(U, q_1, p_1, \ldots, q_k, p_k, y_1, \ldots, y_l)$ centered at x such that on U:

$$
\Pi = \sum_{i=1}^{k}\frac{\partial}{\partial q_i}\wedge\frac{\partial}{\partial p_i}+\sum_{1\le i<j\le l}\phi_{i,j}\frac{\partial}{\partial y_i}\wedge\frac{\partial}{\partial y_j},
$$

where the functions $\phi_{i,j}$ vanish at x, and $2k + l = 2n$. Hence by assumption, l is even with $l > 2$. Now note that

$$
\Pi^n = F\frac{\partial}{\partial q_1}\wedge\frac{\partial}{\partial p_1}\wedge\cdots\wedge\frac{\partial}{\partial q_k}\wedge\frac{\partial}{\partial p_k}\wedge\frac{\partial}{\partial y_1}\wedge\cdots\wedge\frac{\partial}{\partial y_l},
$$

where F is a homogeneous polynomial of degree $l/2$ in the variables $\phi_{i,j}$ for $1 \le i < j \le l$. Under the assumption that $l > 2$, we have that $d_xF = 0$. Indeed, for convenience we rename the variables $\phi_{i,j}$ as $z_1, \ldots, z_m$ where $m = l(l-1)/2$, and then we have

$$
F = \sum_{(i_1,\ldots,i_{l/2})\in I} a_{i_1,\ldots,i_{l/2}} z_{i_1}\cdots z_{i_{l/2}},
$$

for some index set $I \subset \{1, \ldots, m\}^{l/2}$ and constants $a_{i_1,\ldots,i_{l/2}}$. Then

$$
d_xF = \sum_{(i_1,\ldots,i_{l/2})\in I}\sum_{j=1}^{l/2} a_{i_1,\ldots,i_{l/2}} z_{i_1}(x)\cdots z_{i_{j-1}}(x)\, d_x z_{i_j}\, z_{i_{j+1}}(x)\cdots z_{i_{l/2}}(x) = 0,
$$

since all $z_k(x)$ are zero for $k = 1, \ldots, m$. So we run into a contradiction with the fact that Π^n is transverse to the zero section at x. This shows that $\mathrm{rank}(\Pi_x) = 2n - 2$. □

As announced before, we obtain the following coordinate expression for log-symplectic structures near their singular loci. This seems to be a well-known result [Cav] [GMP2], but a complete proof is not given anywhere. The last change of coordinates we apply in the proof below is suggested in [GMP2].

Theorem 3.2.2. *Let (M^{2n}, Z, Π) be a log-symplectic manifold and let $x \in Z$. Then there exist coordinates $(U, x_1, y_1, \ldots, x_n, y_n)$ around x such that on U, the hypersurface Z is locally defined by $y_1 = 0$ and*

$$\Pi = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}. \tag{3.3}$$

Proof. The splitting theorem and Lemma 3.2.1 give coordinates $(V, q_1, p_1, \ldots, q_{n-1}, p_{n-1}, y_1, y_2)$ centered at x such that on V:

$$\Pi = \sum_{i=1}^{n-1} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i} + \phi(y_1, y_2) \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2},$$

where ϕ vanishes at x. Even more is true: since Π has rank $2n - 2$ at points of Z and has full rank elsewhere, we have that $\phi^{-1}(0) = Z \cap V$. That is, $Z \cap V$ is given by $\phi = 0$. Moreover, as Π^n is transverse to the zero section, we have that $d_x \phi \neq 0$. Hence $\partial\phi/\partial y_1$ or $\partial\phi/\partial y_2$ must be nonzero at x. Switching the roles of y_1 and y_2 if necessary, we can assume that $\partial\phi/\partial y_2$ is nonzero at x. Now consider the map

$$(q_1, p_1, \ldots, q_{n-1}, p_{n-1}, y_1, y_2) \mapsto \big(\widetilde{q_1}, \widetilde{p_1}, \ldots, \widetilde{q_{n-1}}, \widetilde{p_{n-1}}, \widetilde{y_1}, \phi(y_1, y_2)\big),$$

where $\widetilde{q_i} = q_i$ and $\widetilde{p_i} = p_i$ for $i = 1, \ldots, n-1$ and $\widetilde{y_1} = y_1$. This is a change of coordinates around x: its Jacobian determinant is

$$\det \left[\begin{array}{ccc|cc} \multicolumn{3}{c|}{I_{(2n-2)\times(2n-2)}} & \multicolumn{2}{c}{0_{(2n-2)\times 2}} \\ \hline 0 & \ldots & 0 & 1 & 0 \\ 0 & \ldots & 0 & \frac{\partial\phi}{\partial y_1} & \frac{\partial\phi}{\partial y_2} \end{array}\right] = \frac{\partial\phi}{\partial y_2},$$

which by continuity of $\partial\phi/\partial y_2$ is nonzero on some smaller neighborhood $U \subset V$ of x. So $(\widetilde{q_1}, \widetilde{p_1}, \ldots, \widetilde{q_{n-1}}, \widetilde{p_{n-1}}, \widetilde{y_1}, \phi)$ are coordinates on U. The coordinate vector fields transform correspondingly:

$$\begin{cases} \dfrac{\partial}{\partial q_i} = \dfrac{\partial}{\partial \widetilde{q_i}} \text{ for } i = 1, \ldots, n-1 \\ \dfrac{\partial}{\partial p_i} = \dfrac{\partial}{\partial \widetilde{p_i}} \text{ for } i = 1, \ldots, n-1 \\ \dfrac{\partial}{\partial y_1} = \dfrac{\partial}{\partial \widetilde{y_1}} + \dfrac{\partial\phi}{\partial y_1} \dfrac{\partial}{\partial\phi} \\ \dfrac{\partial}{\partial y_2} = \dfrac{\partial\phi}{\partial y_2} \dfrac{\partial}{\partial\phi}. \end{cases}$$

Hence, in these new coordinates on U, Π is given by

$$\Pi = \sum_{i=1}^{n-1} \frac{\partial}{\partial \widetilde{q_i}} \wedge \frac{\partial}{\partial \widetilde{p_i}} + \phi \frac{\partial\phi}{\partial y_2} \frac{\partial}{\partial \widetilde{y_1}} \wedge \frac{\partial}{\partial\phi},$$

where $\partial\phi/\partial y_2$ is non-vanishing on U. At last, we change coordinates once more:

$$(\widetilde{q_1}, \widetilde{p_1}, \ldots, \widetilde{q_{n-1}}, \widetilde{p_{n-1}}, \widetilde{y_1}, \phi) \mapsto (\widetilde{q_1}, \widetilde{p_1}, \ldots, \widetilde{q_{n-1}}, \widetilde{p_{n-1}}, \xi, \widetilde{\phi}),$$

where $\widetilde{\phi} = \phi$ and

$$\xi := \int \frac{1}{\frac{\partial\phi}{\partial y_2}} d\widetilde{y_1}.$$

These are indeed new coordinates on U, since

$$\det \begin{bmatrix} \frac{\partial\xi}{\partial\widetilde{y_1}} & \frac{\partial\xi}{\partial\widetilde{\phi}} \\ \frac{\partial\phi}{\partial\widetilde{y_1}} & \frac{\partial\phi}{\partial\widetilde{\phi}} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{\frac{\partial\phi}{\partial y_2}} & \frac{\partial\xi}{\partial\widetilde{\phi}} \\ \frac{\partial\phi}{\partial y_1} - \frac{\partial\phi}{\partial y_1}\frac{\partial\phi}{\partial\phi} & 1 \end{bmatrix} = \det \begin{bmatrix} \frac{1}{\frac{\partial\phi}{\partial y_2}} & \frac{\partial\xi}{\partial\widetilde{\phi}} \\ 0 & 1 \end{bmatrix} = \frac{1}{\frac{\partial\phi}{\partial y_2}}$$

is non-vanishing on U. The coordinate vector fields transform as

$$\begin{cases} \frac{\partial}{\partial\widetilde{y_1}} = \frac{1}{\frac{\partial\phi}{\partial y_2}}\frac{\partial}{\partial\xi} \\ \frac{\partial}{\partial\phi} = \frac{\partial\xi}{\partial\phi}\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\widetilde{\phi}} \end{cases}.$$

Note that $\partial/\partial\widetilde{q_i}$ has the same meaning in both coordinate systems, and the same holds for $\partial/\partial\widetilde{p_i}$. In the coordinates $(U, \widetilde{q_1}, \widetilde{p_1}, \ldots, \widetilde{q_{n-1}}, \widetilde{p_{n-1}}, \xi, \widetilde{\phi})$, we get

$$\Pi = \sum_{i=1}^{n-1} \frac{\partial}{\partial\widetilde{q_i}} \wedge \frac{\partial}{\partial\widetilde{p_i}} + \widetilde{\phi}\frac{\partial}{\partial\xi} \wedge \frac{\partial}{\partial\widetilde{\phi}},$$

which is of the desired form (3.3). Since $\widetilde{\phi}$ is a local defining function for Z on U, this concludes the proof. □

This normal form theorem has some interesting consequences [GMP1] [GMP2].

Corollary 3.2.3. *If (M^{2n}, Z, Π) is a log-symplectic manifold, then Z is a Poisson submanifold of M and the induced Poisson structure on Z is regular of corank one.*

Proof. Let $p \in Z$ and choose coordinates $(U, x_1, y_1, \ldots, x_n, y_n)$ around p as in Theorem 3.2.2. So $U \cap Z$ is given by $y_1 = 0$, and

$$\Pi|_U = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}.$$

In particular,

$$\Pi_p = \sum_{i=2}^{n} \left.\frac{\partial}{\partial x_i}\right|_p \wedge \left.\frac{\partial}{\partial y_i}\right|_p \in \wedge^2 T_p Z. \tag{3.4}$$

This shows that Z is a Poisson submanifold of M. It is clear that the restriction of Π to Z has rank $2n-2$ at all points. This follows immediately from Lemma 3.2.1, but it is also apparent from the expression (3.4). □

Remark 3.2.4. Away from $Z \leftrightarrow \{y_1 = 0\}$, one can invert the bivector

$$\Pi = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

to obtain the differential form

$$\omega = dx_1 \wedge d\log|y_1| + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

This justifies the terminology "log-symplectic": ω acquires a logarithmic singularity along the singular locus Z of Π.

By Corollary 3.2.3, a log-symplectic structure on M gives for free a corank-one Poisson manifold Z with corresponding codimension one symplectic foliation. The following lemma shows another way of constructing corank-one Poisson structures out of log-symplectic structures. It is mentioned in the introduction of [MO2].

Lemma 3.2.5. *Let (M^{2n}, Z, Π) be a log-symplectic manifold. Let X be a Poisson vector field on M that is transverse to the symplectic leaves of Z. Then $\widetilde{\Pi} := \Pi + X \wedge \frac{\partial}{\partial \theta}$ is a corank-one Poisson structure on $M \times S^1$.*

Proof. We first check that $\widetilde{\Pi}$ is Poisson. We have

$$\begin{aligned}
\left[\widetilde{\Pi}, \widetilde{\Pi}\right] &= [\Pi, \Pi] + 2\left[\Pi, X \wedge \frac{\partial}{\partial\theta}\right] + \left[X \wedge \frac{\partial}{\partial\theta}, X \wedge \frac{\partial}{\partial\theta}\right] \\
&= 2\left[\Pi, X \wedge \frac{\partial}{\partial\theta}\right] + \left[X \wedge \frac{\partial}{\partial\theta}, X \wedge \frac{\partial}{\partial\theta}\right] \\
&= 2[\Pi, X] \wedge \frac{\partial}{\partial\theta} - 2X \wedge \left[\Pi, \frac{\partial}{\partial\theta}\right] + \left[X \wedge \frac{\partial}{\partial\theta}, X\right] \wedge \frac{\partial}{\partial\theta} - X \wedge \left[X \wedge \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta}\right] \\
&= -2(\pounds_X \Pi) \wedge \frac{\partial}{\partial\theta} + 2X \wedge (\pounds_{\frac{\partial}{\partial\theta}} \Pi) - \left(\pounds_X \left(X \wedge \frac{\partial}{\partial\theta}\right)\right) \wedge \frac{\partial}{\partial\theta} + X \wedge \left(\pounds_{\frac{\partial}{\partial\theta}} \left(X \wedge \frac{\partial}{\partial\theta}\right)\right) \\
&= -(\pounds_X X) \wedge \frac{\partial}{\partial\theta} \wedge \frac{\partial}{\partial\theta} - X \wedge \left(\pounds_X \frac{\partial}{\partial\theta}\right) \wedge \frac{\partial}{\partial\theta} + X \wedge \left(\pounds_{\frac{\partial}{\partial\theta}} X\right) \wedge \frac{\partial}{\partial\theta} + X \wedge X \wedge \left(\pounds_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta}\right) \\
&= 0,
\end{aligned}$$

where we used that $\pounds_X \Pi = 0$ and that Π nor X depend on θ. To argue that $\widetilde{\Pi}$ is regular of corank-one, we note that

$$\widetilde{\Pi}^n = \Pi^n + n\Pi^{n-1} \wedge X \wedge \frac{\partial}{\partial\theta}.$$

On $(M \setminus Z) \times S^1$, the first term does not vanish. Since X is transverse to the leaves of Z, the second term does not vanish on $Z \times S^1$. As the terms cannot cancel each other, it follows that $\widetilde{\Pi}^n$ is nowhere vanishing, which implies that the rank of $\widetilde{\Pi}$ is $2n$. ☐

This lemma is useful in practice because log-symplectic structures (M, Z, Π) have a convenient class of transverse Poisson vector fields. Indeed, in the next chapter we will show that modular vector fields on (M, Π) are transverse to the symplectic leaves of Z.

Example 3.2.6 ([GMP2])**.** Let (N^{2n+1}, Π) be a regular corank-one Poisson manifold, X a Poisson vector field on N and $f : S^1 \to \mathbb{R}$ a smooth function. The bivector field

$$\widetilde{\Pi} = f(\theta)\frac{\partial}{\partial\theta} \wedge X + \Pi \tag{3.5}$$

is a log-symplectic structure on $S^1 \times N$, provided that the function f vanishes linearly and the vector field X is transverse to the symplectic leaves of N. Indeed, computations similar to those in the proof of Lemma 3.2.5 show that $\widetilde{\Pi}$ is a Poisson structure. Moreover, we have

$$\widetilde{\Pi}^{n+1} = \Pi^{n+1} + (n+1) f(\theta) \frac{\partial}{\partial \theta} \wedge X \wedge \Pi^n = (n+1) f(\theta) \frac{\partial}{\partial \theta} \wedge X \wedge \Pi^n,$$

since Π^{n+1} is a $(2n+2)$-vector field on the $(2n+1)$-dimensional manifold N hence necessarily zero. Since Π is of rank $2n$, we have that Π^n is non-vanishing and since X is transverse to the leaves of N, then also $X \wedge \Pi^n$ is non-vanishing. This can be seen, for instance, by choosing splitting coordinates for Π. Consequently, $\frac{\partial}{\partial \theta} \wedge X \wedge \Pi^n$ is non-vanishing on $S^1 \times N$ and the fact that f vanishes linearly implies that $(S^1 \times N, \widetilde{\Pi})$ is log-symplectic. Its singular locus consists of as many copies of N as f has zeros.

This example is interesting because a slight adaptation of it provides the semilocal model for an orientable log-symplectic structure in a neighborhood of the exceptional hypersurface Z. Indeed, let us replace S^1 by an interval $(-\epsilon, \epsilon)$ with coordinate t and take for $f : (-\epsilon, \epsilon) \to \mathbb{R} : t \mapsto t$ the identity function. Consider the corank-one Poisson structure (Z, Π_Z) induced by an orientable log-symplectic structure (M, Z, Π), and let X be the restriction to Z of a modular vector field on M. Then the expression (3.5) becomes

$$\Pi = t \frac{\partial}{\partial t} \wedge X + \Pi_Z,$$

which is exactly the normal form for Π near Z that we will derive later (see Theorem 5.2.1).

Chapter 4

b-Geometry

Log-symplectic structures are described conveniently in the language of *b*-geometry. Here "b" stands for boundary and refers to the calculus developed by Melrose for differential operators on manifolds with boundary [Mel]. This formalism can easily be adapted to the case of a manifold with a distinguished hypersurface, examples of which are log-symplectic manifolds.

In this chapter, we introduce the *b*-category and its main concepts. As it turns out, log-symplectic structures can be regarded as "symplectic" structures in the *b*-category. This point of view allows us to apply symplectic techniques in the study of log-symplectic structures, leading to extensions of theorems in symplectic geometry to the log-symplectic setting. All these results put log-symplectic structures closer to the symplectic world than to the usually cumbersome Poisson world. This chapter roughly follows [GMP2], complemented by the first three sections of [MO]. Note however that in [GMP2], one works under orientability conditions that we will not impose.

4.1 b-manifolds and b-differential forms

We will first define *b*-manifolds and *b*-maps, which are respectively the objects and the morphisms of the *b*-category. Next, we introduce the *b*-tangent and *b*-cotangent bundle, and the notion of differential forms on *b*-manifolds.

4.1.1 b-manifolds

Definition 4.1.1. A *b*-manifold is a pair (M, Z) consisting of a manifold M and a hypersurface $Z \subset M$. A *b*-map $f : (M_1, Z_1) \to (M_2, Z_2)$ is a smooth map between manifolds $f : M_1 \to M_2$ such that $f^{-1}(Z_2) = Z_1$ and f is transverse to Z_2. That is

$$\mathrm{Im}(d_p f) + T_{f(p)} Z_2 = T_{f(p)} M_2 \quad \text{for all } p \in Z_1.$$

For our purposes, the example to keep in mind here is that of a log-symplectic manifold M with its singular locus $Z \subset M$. In [GMP2], one only considers *b*-manifolds (M, Z) for which both M and Z are orientable, so that one can assume that Z is defined by the vanishing of a smooth function (See Lemma 4.1.2) that is defined in a neighborhood of Z. However, since log-symplectic manifolds need not be orientable (see Example 3.1.5), this restriction is too stringent for us.

Lemma 4.1.2. *Let M be an orientable manifold and $Z \subset M$ a hypersurface. Then Z is orientable if and only if there exists a b-map $f : (U', Z) \to (\mathbb{R}, \{0\})$, where U' is a tubular neighborhood of Z.*

Proof. First assume that Z is orientable. We are asked to find a tubular neighborhood U' of Z and a smooth map $f : U' \to \mathbb{R}$ such that $f^{-1}(0) = Z$ and $d_p f \neq 0$ for all $p \in Z$. Equip M with a Riemannian metric. Consider the normal bundle of Z in M, consisting of the orthogonal complements of the tangent spaces $T_pZ \subset T_pM$:

$$TZ^{\perp} := \{(p, v) : p \in Z, v \in (T_pZ)^{\perp} \subset T_pM\}.$$

We define the normal exponential map $\exp^{\perp}$ by restricting the exponential map to the normal bundle:

$$\exp^{\perp} : U \subset TZ^{\perp} \to M : (p, v) \mapsto \exp_p(v),$$

where U is an open neighborhood of the zero section $Z \subset TZ^{\perp}$. Note that at $(p, 0_p) \in TZ^{\perp}$, the derivative $d_{(p,0_p)} \exp^{\perp} : T_pZ \times T_pZ^{\perp} \to T_pM$ is an isomorphism. Indeed, working in a local trivialization $V \times \mathbb{R}$ near p, let $(z, w) \in T_pZ \times \mathbb{R}$. Consider the curve $t \mapsto (\beta(t), wt)$ in $V \times \mathbb{R}$, where β is a curve in V passing through p at time $t = 0$ with tangent vector z. By the chain rule, we have

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \exp^{\perp}(\beta(t), wt) &= \left.\frac{d}{dt}\right|_{t=0} \exp_{\beta(t)}(wt) = \left.\frac{d}{dx}\right|_{x=0} \exp_{\beta(x)}(0) + \left.\frac{d}{dy}\right|_{y=0} \exp_{\beta(0)}(wy) \\ &= \left.\frac{d}{dx}\right|_{x=0} \beta(x) + \left.\frac{d}{dy}\right|_{y=0} \exp_p(wy) = z + w, \end{aligned}$$

where we used that $\exp_q(0) = q$ and $(d\exp_q)_0 = \mathrm{Id}_{T_qM}$ for $q \in M$. Hence, we have that

$$d_{(p,0_p)} \exp^{\perp} : T_pZ \times T_pZ^{\perp} \to T_pM : (z, w) \mapsto z + w.$$

Counting dimensions, it is enough to show that this map is injective. But injectivity is clear since $T_pM = T_pZ \oplus T_pZ^{\perp}$. By the inverse function theorem, we find a neighborhood $V' \subset TZ^{\perp}$ of the zero section and a neighborhood $U' \subset M$ of Z such that $\exp^{\perp} : V' \to U'$ is a diffeomorphism. Note that $\exp^{\perp}$ takes the zero section of $TZ^{\perp}$ to Z. Now, since M and Z are orientable, the normal bundle is trivial: $TZ^{\perp} \cong Z \times \mathbb{R}$. Denote by $\pi_2 : Z \times \mathbb{R} \to \mathbb{R}$ the projection. We define $f : U' \to \mathbb{R}$ by $f := \pi_2 \circ (\exp^{\perp})^{-1}$. Then f is a submersion, being a composition of submersions, and thus $d_p f \neq 0$ for all $p \in Z$. Moreover, $f^{-1}(0) = \exp^{\perp}(Z \times \{0\}) = Z$.
Conversely, assume we have a b-map $f : (U', Z) \to (\mathbb{R}, \{0\})$. Note that $d_p f \neq 0$ for all $p \in Z$, and that $(df)|_Z$ vanishes on vector fields tangent to Z. Indeed, if $X_p \in T_pZ$ then we have

$$(d_p f)(X_p) = \big(d_p(f|_Z)\big)(X_p) = 0$$

as $f|_Z = 0$. Hence $df|_Z$ trivializes the conormal bundle of Z in M. Dualizing, we get that the normal bundle of Z in M is trivial as well, which along with the fact that M is orientable implies orientability of Z. □

Remark 4.1.3. In the proof of the "only if" implication of Lemma 4.1.2, we could have taken a shortcut by applying the Tubular Neighborhood Theorem 1.3.8, which ensures that a neighborhood of the zero section in $TZ^{\perp}$ is diffeomorphic to a neighborhood of Z through a diffeomorphism taking the zero section to Z. However, since the proof of the Tubular Neighborhood Theorem is omitted, it seemed interesting to construct such a diffeomorphism explicitly for once.

In Lemma 4.1.2, we only managed to find a defining function for Z in a tubular neighborhood of Z. Extending this function to a global defining function on M is a rather delicate issue in general. For instance, consider the b-manifold $(S^1, \{p\})$ where $p \in S^1$ is a point. Then it is not possible to find a smooth function on S^1 that vanishes linearly at p and is non-vanishing elsewhere. Luckily, for log-symplectic manifolds the situation is a lot easier, as the next remark shows.

Remark 4.1.4. Let (M, Z, Π) be a log-symplectic manifold. When M is orientable, a defining function for Z exists automatically by Corollary 3.1.4. This need not be the case when M is not orientable. For consider the b-manifold $(\mathbb{RP}^2, \mathbb{RP}^1)$, which is log-symplectic by Example 3.1.5; then there exists no function $f : \mathbb{RP}^2 \to \mathbb{R}$ for which 0 is a regular value and $f^{-1}(0) = \mathbb{RP}^1$.
First note that $\mathbb{RP}^2 \setminus \mathbb{RP}^1$ is connected. Indeed, assume that $\mathbb{RP}^2 \setminus \mathbb{RP}^1 = U \cup V$ is a separation and denote by $\phi : S^2 \to \mathbb{RP}^2$ the quotient map that identifies antipodal points. Since ϕ is continuous for the quotient topology on $\mathbb{RP}^2$, we get that $\phi^{-1}(U)$ and $\phi^{-1}(V)$ are disjoint open subsets of S^2, with $S^2 \setminus S^1 = \phi^{-1}(U) \cup \phi^{-1}(V)$. Intersecting $\phi^{-1}(U)$ and $\phi^{-1}(V)$ with the open upper hemisphere S^2_+ would give a separation of S^2_+, if both $\phi^{-1}(U) \cap S^2_+$ and $\phi^{-1}(V) \cap S^2_+$ were nonempty. Since S^2_+ is connected, it follows that we may assume that $\phi^{-1}(V) \subset S^2_-$. The same argument applied to S^2_- then yields that $\phi^{-1}(U) = S^2_+$ and $\phi^{-1}(V) = S^2_-$. But then for $x \in \phi^{-1}(U)$, we have that its antipodal point $-x \in \phi^{-1}(V)$. Hence $\phi(x) = \phi(-x) \in U \cap V$, which contradicts that U and V separate $\mathbb{RP}^2 \setminus \mathbb{RP}^1$.
Now assume that $f : \mathbb{RP}^2 \to \mathbb{R}$ is a smooth function with $f^{-1}(0) = \mathbb{RP}^1$. Since $\mathbb{RP}^2 \setminus \mathbb{RP}^1$ is connected, and f is never zero on it, f must have constant sign on $\mathbb{RP}^2 \setminus \mathbb{RP}^1$. Replacing f by $-f$ if necessary, we can assume that $f > 0$ on $\mathbb{RP}^2 \setminus \mathbb{RP}^1$. But then 0 is a global minimum of f, which implies that the derivative of f must vanish at all points of $\mathbb{RP}^1$. In particular, every point of $\mathbb{RP}^1$ is a singular point of f, and 0 is not a regular value of f.

4.1.2 b-tangent and b-cotangent bundles

Definition 4.1.5. Let (M, Z) be a b-manifold. A b-vector field on (M, Z) is a vector field on M which is tangent to Z at each point $p \in Z$. We denote the set of b-vector fields by ${}^b\mathfrak{X}(M)$.

Example 4.1.6. Let (M, Z, Π) be a log-symplectic manifold. Since $Z \subset M$ is a Poisson submanifold, every hamiltonian vector field on M is tangent to Z at points $p \in Z$. Consequently, hamiltonian vector fields on M are b-vector fields on (M, Z).

Note that a vector field $X \in \mathfrak{X}(M)$ is a b-vector field on (M, Z) if and only if around every $p \in Z$, one can find adapted coordinates $(U, x_1, \ldots, x_n)$ such that $Z \cap U$ is defined by $x_1 = 0$ and

$$X|_U = f_1 x_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + \cdots + f_n \frac{\partial}{\partial x_n}$$

for unique smooth functions $f_1, \ldots, f_n \in C^\infty(U)$. So the set of b-vector fields is a locally free $C^\infty(M)$-module, with local bases

$$\left\{ x_1 \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right\} \qquad \text{near } Z \leftrightarrow \{x_1 = 0\}.$$

$$\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\} \qquad \text{away from } Z.$$

Recall that the Serre-Swan Theorem asserts that the category of smooth vector bundles over M is equivalent with the category of locally free $C^\infty(M)$-modules of finite rank [Tay, Proposition 7.6.5]. We use the following light version.

Theorem 4.1.7 (Serre-Swan)**.** *Let M be a smooth manifold. There is a $1:1$ correspondence between smooth vector bundles over M and locally free $C^\infty(M)$-modules of finite rank.*

Proof. Suppose $\Pi : E \to M$ is a vector bundle of rank k. Cover M in opens $\{U_i\}_{i \in I}$ that constitute local trivializations of E, and assign to each open U_i the $C^\infty(U_i)$-module $\mathcal{M}|_{U_i}$ consisting of sections $U_i \to E$. The modules $\mathcal{M}|_{U_i}$ are free: through local trivialization, sections $U_i \to U_i \times \mathbb{R}^k$ compose with the first projection to give the identity map on U_i. Hence a section

$U_i \to U_i \times \mathbb{R}^k$ corresponds with a function $U_i \to \mathbb{R}^k$, and the latter is just a list of k functions $U_i \to \mathbb{R}$. Hence $\mathcal{M}|_{U_i} \cong C^\infty(U_i)^k$, and $\mathcal{M}$ is a locally free $C^\infty(M)$-module of rank k. Conversely, let $\mathcal{M}$ be a locally free $C^\infty(M)$-module of rank k. For $x \in M$, define

$$\mathcal{F}_x := \frac{\mathcal{M}|_{U_x}}{I_x \mathcal{M}|_{U_x}},$$

where I_x is the ideal of functions that vanish at x and that are defined on a neighborhood U_x of x so that $\mathcal{M}|_{U_x}$ is free. Then $\mathcal{F}_x$ is a k-dimensional vector space: if $\{m_1, \ldots, m_k\}$ is a basis for $\mathcal{M}|_{U_x}$ over $C^\infty(U_x)$, then $\{\overline{m_1}, \ldots, \overline{m_k}\}$ is an $\mathbb{R}$-basis for $\mathcal{F}_x$. Let us first check linear independence. Assume $r_1\overline{m_1} + \cdots + r_k\overline{m_k} = 0$ for some $r_1, \ldots, r_k \in \mathbb{R}$. Then $\overline{r_1 m_1 + \cdots + r_k m_k} = 0$ in $\mathcal{F}_x$, which implies that $r_1m_1 + \cdots + r_k m_k \in I_x\mathcal{M}|_{U_x}$. Hence $r_1m_1+\cdots+r_km_k = f_1\xi_1 + \cdots + f_m\xi_m$ for some $f_j \in I_x$ and $\xi_j \in \mathcal{M}|_{U_x}$. Expressing the ξ_j in terms of $m_1, \ldots, m_k$ gives that

$$r_1m_1 + \cdots + r_km_k = g_1m_1 + \cdots + g_km_k,$$

for some $g_i \in C^\infty(U_x)$ with $g_i(x) = 0$. But then $(g_1 - r_1)m_1 + \cdots + (g_k - r_k)m_k = 0$, and since $\{m_1, \ldots, m_k\}$ is free over $C^\infty(U_x)$, this implies that $g_i = r_i$ on U_x. In particular, $g_i(x) = 0 = r_i$. This shows that $\{\overline{m_1}, \ldots, \overline{m_k}\}$ is an $\mathbb{R}$-linearly independent set. Next, $\{\overline{m_1}, \ldots, \overline{m_k}\}$ generates $\mathcal{F}_x$ over $\mathbb{R}$ since

$$\overline{f_1m_1 + \cdots + f_km_k} = \overline{f_1(x)m_1 + \cdots + f_k(x)m_k} = f_1(x)\overline{m_1} + \cdots + f_k(x)\overline{m_k},$$

as the functions $f_i - f_i(x)$ vanish at x. Put $E := \cup_{x\in M}\mathcal{F}_x$ and let $\Pi : E \to M$ be defined by $\Pi(\mathcal{F}_x) = x$. If $U \subset M$ is open so that $\mathcal{M}|_U$ is free (with basis $\{m_1, \ldots, m_k\}$), then we have an isomorphism $\phi_U : \mathcal{M}|_U \to C^\infty(U)^k : f_1m_1 + \cdots + f_km_k \mapsto (f_1, \ldots, f_k)$. This gives rise to a map

$$\psi_U : E|_U \to U \times \mathbb{R}^k : \overline{m} \in \mathcal{F}_x \mapsto \big(x, \phi_U(m)(x)\big),$$

which is bijective and an isomorphism in the fibers $\mathcal{F}_x \to \{x\} \times \mathbb{R}^k$. Indeed, the map $(\psi_U)|_{F_x}$ is clearly linear, and it is well-defined and injective since

$$\overline{m} = \overline{m'} \Leftrightarrow m - m' \in I_x\mathcal{M}|_U \Leftrightarrow \phi_U(m - m')(x) = 0.$$

By dimension reasons, ψ_U is a fiberwise isomorphism. Moreover, if $U \cap V \neq \emptyset$, then we have

$$\psi_U \circ \psi_V^{-1} : (U \cap V) \times \mathbb{R}^k \to (U \cap V) \times \mathbb{R}^k : (x, v) \mapsto \big(x, \big(\phi_U \circ \phi_V^{-1}\big)(x)(v)\big),$$

where we consider $\phi_U \circ \phi_V^{-1}$ as an invertible $(k \times k)$ matrix with entries in $C^\infty(U \cap V)$. It is well-known that the data now obtained determine a smooth vector bundle structure on $\Pi : E \to M$ [Lee, Lemma 10.6]. Moreover, $\mathcal{M} \cong \Gamma(E)$ as locally free $C^\infty(M)$-modules, for if $\mathcal{M}|_U$ is free on basis $\{m_1, \ldots, m_k\}$ and $f = (f_1, \ldots, f_k) : U \to \mathbb{R}^k$ is a local section of E, then we have an isomorphism

$$\Gamma(E)|_U \to \mathcal{M}|_U : f \mapsto f_1m_1 + \cdots + f_km_k.$$

□

So there exists a unique vector bundle over M whose sections are the b-vector fields.

Definition 4.1.8. Let (M, Z) be a b-manifold. The b-tangent bundle ${}^bTM$ is the vector bundle whose sections are the b-vector fields on (M, Z).

Note that

$$ {}^bT_pM = \begin{cases} T_pZ \oplus \left\langle \left(x_1\frac{\partial}{\partial x_1}\right)\Big|_p \right\rangle & \text{if } p \in Z \leftrightarrow \{x_1 = 0\} \\ T_pM & \text{if } p \notin Z \end{cases} , \tag{4.1} $$

where it is worth noting that $x_1\frac{\partial}{\partial x_1}$ is nowhere vanishing as a b-vector field, whereas it vanishes at points of Z when considered as a vector field.

We now show that the b-tangent bundle ${}^bTM$ has a natural Lie algebroid structure. Recall that a Lie algebroid over M is a triple $(\mathcal{A}, [\cdot,\cdot]_\mathcal{A}, \rho_\mathcal{A})$, where $\mathcal{A} \to M$ is a vector bundle, $\rho_\mathcal{A} : \mathcal{A} \to TM$ is a bundle map and $\big(\Gamma(\mathcal{A}), [\cdot,\cdot]_\mathcal{A}\big)$ is a Lie algebra, such that

$$ [a, fb]_\mathcal{A} = \pounds_{\rho_\mathcal{A}(a)} f \cdot b + f \cdot [a,b]_\mathcal{A}, \tag{4.2} $$

for $a, b \in \Gamma(\mathcal{A})$ and $f \in C^\infty(M)$. The map $\rho_\mathcal{A}$ is called the anchor.

Lemma 4.1.9 ([Lee])**.** *Let M^n be a manifold and $Z \subset M$ a k-dimensional submanifold. If $V, W \in \mathfrak{X}(M)$ are tangent to Z, then the same holds for their Lie bracket $[V,W]$.*

Proof. Let $i : Z \hookrightarrow M$ denote the inclusion. We first show that there exist smooth vector fields $\overline{V}, \overline{W}$ on Z such that $\overline{V}$ is i-related with V and $\overline{W}$ is i-related with W. The fact that V is tangent to Z means that V_p lies in the image of d_pi for each $p \in Z$. Thus for each $p \in Z$, there exists a vector $\overline{V}_p \in T_pZ$ such that $d_pi\left(\overline{V}_p\right) = V_p$. Since d_pi is injective, this vector $\overline{V}_p$ is even unique. It remains to show that $\overline{V}$ is smooth. Choose adapted coordinates around $p \in Z$ so that locally Z is given by $x_{k+1} = \cdots = x_n = 0$. If $V = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$ in these coordinates, then by construction $\overline{V} = \sum_{i=1}^k f_i \frac{\partial}{\partial x_i}$, which is clearly smooth. We proceed similarly to construct $\overline{W}$. Lemma 8.3.2 in the appendix now implies that $[V,W]$ and $[\overline{V}, \overline{W}]$ are i-related. This implies that $[V,W]$ is tangent to Z. □

We now define a Lie algebroid structure on ${}^bTM$ as follows. Since the inclusion map ${}^b\mathfrak{X}(M) \hookrightarrow \mathfrak{X}(M)$ is $C^\infty(M)$-linear, it comes from a vector bundle map $\rho : {}^bTM \to TM$, which we define to be the anchor of ${}^bTM$. Next, by the above lemma, we can restrict the Lie bracket on $\mathfrak{X}(M)$ to ${}^b\mathfrak{X}(M) = \Gamma({}^bTM)$. The identity (4.2) follows automatically from the defining properties of the Lie bracket (see for instance [Lee, Proposition 8.28]).

We can reinterpret the observation 4.1 in terms of the anchor map $\rho : {}^bTM \to TM$. Over $M \setminus Z$, the map ρ is the identity map. Restricting ρ to Z gives a bundle epimorphism

$$ \psi : {}^bTM|_Z \to TZ, \tag{4.3} $$

in the fiber above $p \in Z \leftrightarrow \{x_1 = 0\}$ given by

$$ {}^bT_pM \to T_pZ : a_1\left(x_1\frac{\partial}{\partial x_1}\right)\bigg|_p + a_2\,\frac{\partial}{\partial x_2}\bigg|_p + \cdots + a_n\,\frac{\partial}{\partial x_n}\bigg|_p \mapsto a_2\,\frac{\partial}{\partial x_2}\bigg|_p + \cdots + a_n\,\frac{\partial}{\partial x_n}\bigg|_p . \tag{4.4} $$

A priori, it may seem that the definition of this map depends on the chosen coordinates, but the proof of the lemma below shows that this is not the case. For suppose $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are both coordinate systems around $p \in Z$ such that x_1 and y_1 are defining functions for Z. Assume that

$$ a_1\left(x_1\frac{\partial}{\partial x_1}\right)\bigg|_p + a_2\,\frac{\partial}{\partial x_2}\bigg|_p + \cdots + a_n\,\frac{\partial}{\partial x_n}\bigg|_p = b_1\left(y_1\frac{\partial}{\partial y_1}\right)\bigg|_p + b_2\,\frac{\partial}{\partial y_2}\bigg|_p + \cdots + b_n\,\frac{\partial}{\partial y_n}\bigg|_p . $$

The argument below shows that

$$ \left(x_1\frac{\partial}{\partial x_1}\right)\bigg|_p = \left(y_1\frac{\partial}{\partial y_1}\right)\bigg|_p , $$

and then $a_1 = b_1$ by 4.1. Hence we get

$$a_2 \left.\frac{\partial}{\partial x_2}\right|_p + \cdots + a_n \left.\frac{\partial}{\partial x_n}\right|_p = b_2 \left.\frac{\partial}{\partial y_2}\right|_p + \cdots + b_n \left.\frac{\partial}{\partial y_n}\right|_p .$$

Lemma 4.1.10. *The kernel of the map* (4.3) *is a line bundle* $\mathbb{L}_Z$ *with canonical non-vanishing section.*

Proof. From the expression (4.4), it is clear that the map (4.3) has a one dimensional kernel at each point $p \in Z$, which is spanned by $\left.\left(x_1 \frac{\partial}{\partial x_1}\right)\right|_p$ if Z is locally given by $x_1 = 0$ in adapted coordinates. In particular, the map (4.3) is of constant corank equal to 1, which implies that its kernel is a line subbundle of ${}^bTM$ [Lee, Theorem 10.34]. We now show that the b-vector field $x_1 \frac{\partial}{\partial x_1}$ (where $Z \leftrightarrow \{x_1 = 0\}$) at points of Z is independent of choice of coordinates. So assume $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are coordinate systems around $p \in Z$ so that both x_1 and y_1 are locally defining functions for Z. Then we must have that $y_1 = hx_1$ for some non-vanishing function h defined near p. This implies that

$$\frac{\partial}{\partial x_1} = \frac{\partial(hx_1)}{\partial x_1}\frac{\partial}{\partial y_1} + \sum_{j=2}^{n} \frac{\partial y_j}{\partial x_1}\frac{\partial}{\partial y_j} = \left(\frac{\partial h}{\partial x_1}\frac{1}{h}\right) y_1 \frac{\partial}{\partial y_1} + h\frac{\partial}{\partial y_1} + \sum_{j=2}^{n} \frac{\partial y_j}{\partial x_1}\frac{\partial}{\partial y_j},$$

hence

$$\left.\left(x_1\frac{\partial}{\partial x_1}\right)\right|_p = \left.\left(\frac{\partial h}{\partial x_1}\frac{1}{h}x_1\right)\right|_p \left.\left(y_1\frac{\partial}{\partial y_1}\right)\right|_p + \left.\left(y_1\frac{\partial}{\partial y_1}\right)\right|_p + \sum_{j=2}^{n} \left.\left(x_1\frac{\partial y_j}{\partial x_1}\right)\right|_p \left.\frac{\partial}{\partial y_j}\right|_p = \left.\left(y_1\frac{\partial}{\partial y_1}\right)\right|_p .$$

It follows that we can construct a global canonical trivialization ξ of $\mathbb{L}_Z$, which is locally given by $x_1 \frac{\partial}{\partial x_1}$ in any adapted chart $(x_1, \ldots, x_n)$ which expresses Z locally as $x_1 = 0$. □

Definition 4.1.11. This non-vanishing section ξ of $\mathbb{L}_Z$ is the normal b-vector field of (M, Z).

Remark 4.1.12. In a coordinate-free way, the normal b-vector field ξ is locally given by $(fv)|_Z$, where f is locally defining for Z and v is a vector field such that $df(v) = 1$. It is not hard to see that this indeed defines a non-vanishing section of $\mathbb{L}_Z$ and computations similar to those in previous lemma show that the definition is independent of the choice of f and v.

Definition 4.1.13. The b-cotangent bundle of (M, Z) is the vector bundle ${}^bT^*M$ dual to ${}^bTM$.

Note that, at points $p \in M \setminus Z$, we have that ${}^bT_p^*M = ({}^bT_pM)^* = (T_pM)^* = T_p^*M$ is the ordinary cotangent space. At points $p \in Z$, the map $\psi_p : {}^bT_pM \to T_pZ$ from (4.3) is surjective. Hence its dual map $\psi_p^* : T_p^*Z \to {}^bT_p^*M$ is injective. The image of ψ_p^* is

$$\langle \xi_p \rangle^0 := \{\alpha \in {}^bT_p^*M : \ \alpha(\xi_p) = 0\}.$$

Indeed, let $\beta \in T_p^*Z$. Then $\psi_p^*(\beta)(\xi_p) = \beta(\psi_p(\xi_p)) = 0$ since $\xi_p \in \mathrm{Ker}(\psi_p)$, which shows that $\mathrm{Im}(\psi_p^*) \subset \langle \xi_p \rangle^0$. Since ψ_p^* is injective, we have that $\dim(\mathrm{Im}(\psi_p^*)) = \dim(T_p^*Z) = \dim(\langle \xi_p \rangle^0)$ and thus

$$T_p^*Z \cong \mathrm{Im}(\psi_p^*) = \langle \xi_p \rangle^0. \tag{4.5}$$

Next, let $(x_1, \ldots, x_n)$ be coordinates around p so that Z is locally defined by $x_1 = 0$. Away from Z, we have a well-defined one form $\frac{dx_1}{x_1}$. Its pairing with any b-vector field extends smoothly over Z, since

$$\left\langle f_1 x_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} + \cdots + f_n \frac{\partial}{\partial x_2}, \frac{dx_1}{x_1} \right\rangle = f_1.$$

It follows that $\frac{dx_1}{x_1}$ has a smooth extension over $Z \leftrightarrow \{x_1 = 0\}$ as a section of ${}^bT^*M$, which we will still denote by $\frac{dx_1}{x_1}$ by slight abuse of notation. Moreover, as $\left(\frac{dx_1}{x_1}\right)_p(\xi_p) = 1$, we have $\left(\frac{dx_1}{x_1}\right)_p \notin \langle \xi_p \rangle^0$ and hence we conclude

$$
{}^bT_p^*M = \begin{cases} T_p^*Z \oplus \left\langle \left(\frac{dx_1}{x_1}\right)_p \right\rangle & \text{if } p \in Z \leftrightarrow \{x_1 = 0\} \\ T_p^*M & \text{if } p \notin Z \end{cases} . \tag{4.6}
$$

4.1.3 b-differential forms

Decompositions and the b-de Rham differential

Definition 4.1.14. Let (M, Z) be a b-manifold. For each $k \in \mathbb{N}$, we denote by ${}^b\Omega^k(M)$ the space of b-de Rham k-forms, which are the sections of the vector bundle $\wedge^k({}^bT^*M)$.

We can view differential forms on M as b-forms on (M, Z) by pulling them back under the anchor map. Indeed, first note that $\rho : {}^bTM \to TM$ is an isomorphism on $M \setminus Z$, hence so is its dual map $\rho^* : T^*M \to {}^bT^*M$. Then also the induced map $\rho^* : \wedge^k(T^*M) \to \wedge^k({}^bT^*M)$ is an isomorphism on $M \setminus Z$, and since this set is dense in M, it follows that on the level of sections, the map $\rho^* : \Omega^k(M) \to {}^b\Omega^k(M)$ is injective. Concretely, given $\mu \in \Omega^k(M)$, we interpret it as an element of ${}^b\Omega^k(M)$ by the rules

$$
\begin{cases} \mu_p \in \bigwedge^k(T_p^*M) = \bigwedge^k({}^bT_p^*M) & \text{at } p \in M \setminus Z \\ \mu_p = (i^*\mu)_p \in \bigwedge^k(T_p^*Z) \subset \bigwedge^k({}^bT_p^*M) & \text{at } p \in Z \end{cases} , \tag{4.7}
$$

where $i : Z \hookrightarrow M$ is the inclusion map.

Typically, b-differential forms on (M, Z) explode near Z. Those that vanish at Z are in fact honest de Rham forms.

Lemma 4.1.15. *Let (M, Z) be a b-manifold and $\omega \in {}^b\Omega^k(M)$ a b-de Rham k-form. If $\omega|_Z = 0$, then $\omega \in \Omega^k(M)$.*

Proof. Choose coordinates $(x_1, \dots, x_n)$ around $p \in Z$ so that Z is locally given by $x_1 = 0$. In these coordinates, we write

$$
\omega = \sum_{1 < i_2 < \cdots < i_k \leq n} f_{i_2,\dots,i_k} \frac{dx_1}{x_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} + \sum_{1 < i_1 < \cdots < i_k \leq n} g_{i_1,\dots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.
$$

Since ω vanishes on Z, the same must hold for the functions $f_{i_2,\dots,i_k}$ and $g_{i_1,\dots,i_k}$. In particular, we find functions $h_{i_2,\dots,i_k}$ defined near p so that $f_{i_2,\dots,i_k} = x_1 h_{i_2,\dots,i_k}$. It follows that away from Z, we can write

$$
\begin{aligned}
\omega &= \sum_{1 < i_2 < \cdots < i_k \leq n} x_1 h_{i_2,\dots,i_k} \frac{dx_1}{x_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} + \sum_{1 < i_1 < \cdots < i_k \leq n} g_{i_1,\dots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&= \sum_{1 < i_2 < \cdots < i_k \leq n} h_{i_2,\dots,i_k} dx_1 \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} + \sum_{1 < i_1 < \cdots < i_k \leq n} g_{i_1,\dots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},
\end{aligned}
$$

which extends smoothly over Z as a de Rham k-form whose pullback to Z vanishes. □

We now describe a suitable decomposition of b-forms. Fix a tubular neighborhood $p : E \to Z$ of Z and choose a metric g on E with corresponding distance function $x \mapsto \|x\|$. Construct a function $\lambda : M \setminus Z \to (0, \infty)$ satisfying $\lambda(x) = \|x\|$ for $x \in E$ with $\|x\| \leq 1/2$, and $\lambda \equiv 1$ on $M \setminus \{x \in E : \|x\| < 1\}$. For details on this construction, see the appendix. We now claim:

Lemma 4.1.16. *Let (M, Z) be a b-manifold and $\omega \in {}^b\Omega^k(M)$ a b-form. We can decompose*

$$\omega = \alpha + d\log(\lambda) \wedge p^*(\theta), \tag{4.8}$$

for some $\theta \in \Omega^{k-1}(Z)$ and $\alpha \in \Omega^k(M)$.

Proof. Let us first show that the differential form $d\log(\lambda)$ on $M \setminus Z$ extends smoothly over Z as a b-form. Consider a local trivialization $\psi : p^{-1}(U) \to U \times \mathbb{R}$ of NZ with coordinate t in the fibers, and let $(x_1, \ldots, x_{n-1})$ be coordinates on $U \subset Z$. Close enough to Z, we then have

$$\begin{aligned} \lambda(x_1, \ldots, x_{n-1}, t) &= \sqrt{g\big(t \cdot \psi^{-1}(x_1, \ldots, x_{n-1}, 1), t \cdot \psi^{-1}(x_1, \ldots, x_{n-1}, 1)\big)} \\ &= |t|\sqrt{g\big(\psi^{-1}(x_1, \ldots, x_{n-1}, 1), \psi^{-1}(x_1, \ldots, x_{n-1}, 1)\big)} \\ &:= |t| h(x_1, \ldots, x_{n-1}), \end{aligned} \tag{4.9}$$

where h is smooth and strictly positive since g is positive definite. Hence

$$d\log(\lambda) = d\log(|t|) + d\log(h) = \frac{dt}{t} + d\log(h), \tag{4.10}$$

and we already argued above that dt/t extends over Z as a b-form. Hence the same holds for $d\log(\lambda)$. Now, in the coordinates $(x_1, \ldots, x_{n-1}, t)$, any b-form can be written as a smooth combination of $dt/t, dx_1, \ldots, dx_{n-1}$. Hence by the relation (4.10), b-forms can equally well be written as combinations of $d\log(\lambda), dx_1, \ldots, dx_{n-1}$. It follows that around any $p \in Z$, we can find a neighborhood V so that $\omega|_V$ can be written as

$$\omega|_V = \alpha_V \wedge d\log(\lambda)|_V + \beta_V,$$

for some $\alpha_V \in \Omega^{k-1}(V)$ and $\beta_V \in \Omega^k(V)$. Using a partition of unity subordinate to these opens, we obtain an open neighborhood $O \subset E$ of Z and $\alpha \in \Omega^{k-1}(O), \beta \in \Omega^k(O)$ so that

$$\omega|_O = \alpha \wedge d\log(\lambda)|_O + \beta. \tag{4.11}$$

Now consider the open cover $\{O, M \setminus Z\}$ of M and let $\{\gamma, \delta\}$ be a partition of unity subordinate to this cover. Note that γ is supported inside O, and that $\gamma|_Z \equiv 1$. Consider the globally defined b-form

$$\omega - p^*(\alpha|_Z) \wedge d\log(\lambda) - \gamma p^*(\beta|_Z)$$

and note that

$$\begin{aligned} \big[\omega - p^*(\alpha|_Z) \wedge d\log(\lambda) - \gamma p^*(\beta|_Z)\big]|_Z &= \omega|_Z - (p|_Z)^*(\alpha|_Z) \wedge d\log(\lambda)|_Z - \gamma|_Z (p|_Z)^*(\beta|_Z) \\ &= \omega|_Z - \alpha|_Z \wedge d\log(\lambda)|_Z - \beta|_Z \\ &= 0 \end{aligned}$$

by equation (4.11). By Lemma 4.1.15, we conclude that $\omega - p^*(\alpha|_Z) \wedge d\log(\lambda) - \gamma p^*(\beta|_Z)$ is an honest de Rham k-form, which we call $\eta \in \Omega^k(M)$. It follows that

$$\omega = p^*(\alpha|_Z) \wedge d\log(\lambda) + \gamma p^*(\beta|_Z) + \eta,$$

which is of the desired form (4.8). $\square$

Moreover, for a fixed tubular neighborhood and distance function λ, we have that θ and α in equation (4.8) are unique. Indeed, the decomposition (4.6) in combination with equation (4.10) gives that for $q \in Z \leftrightarrow \{t=0\}$:

$$^bT_q^*M = T_q^*Z \oplus \left\langle \left(\frac{dt}{t}\right)_q \right\rangle = T_q^*Z \oplus \left\langle (d\log(\lambda))_q \right\rangle .$$

Hence,

$$\wedge^k \left({}^bT_q^*M\right) = \left[\wedge^k T_q^*Z\right] \oplus \left[\wedge^{k-1} T_q^*Z \wedge \left\langle (d\log(\lambda))_q \right\rangle\right].$$

Since at $q \in Z$, α_q and $\left(p^*(\theta)\right)_q$ have to be interpreted as elements of $\wedge^k T_q^*Z$ and $\wedge^{k-1}T_q^*Z$ by the conventions (4.7), it follows that the pullbacks of α and $p^*(\theta)$ to Z are unique. Thus, denoting by $i : Z \hookrightarrow M$ the inclusion, we have that $i^*(p^*(\theta)) = (p \circ i)^*(\theta) = \theta$ is unique. Uniqueness of θ then also implies uniqueness of α.

Note that θ is even completely independent of the choice of tubular neighborhood and distance function λ. Indeed, if ξ is the normal b-vector field of (M, Z), then we have

$$\iota_\xi \left(\omega|_Z\right) = \iota_\xi \left(\alpha|_Z + d\log(\lambda)|_Z \wedge \theta\right) = \theta.$$

This is true since for $q \in Z$ we have $\iota_{\xi_q}\alpha_q = \iota_{\xi_q}\theta_q = 0$ (keeping in mind the conventions (4.7) and that $T_q^*Z = \langle \xi_q \rangle^0$ by (4.5)) and $\iota_{\xi_q}(d\log(\lambda))_q = 1$ (Use (4.10) and note that $\iota_{\xi_q}(d\log(h))_q = 0$ as before, whereas $(dt/t)_q(\xi_q) = 1$ as stated in the line above (4.6)). The differential form α however does depend on these choices. Suppose we have distance functions λ and λ', associated with tubular neighborhoods $p : E \to Z$ and $p' : E' \to Z$ respectively. Then we have that λ and λ' differ by a smooth factor $g \in C^\infty(M)$ that is strictly positive. Indeed, although λ and λ' fail to be smooth at points of Z, the function g will be smooth: as in equation (4.9), we can write locally near Z

$$\lambda(x_1, \ldots, x_{n-1}, t) = |t| h(x_1, \ldots, x_{n-1}) \qquad \text{and} \qquad \lambda'(x_1, \ldots, x_{n-1}, t) = |t| h'(x_1, \ldots, x_{n-1}),$$

for smooth functions h, h' that are strictly positive. It follows that we can write $g = e^f$ for some smooth function $f \in C^\infty(M)$ and get $\lambda' = e^f \lambda$ on M. Hence

$$d\log(\lambda') = d\log(\lambda) + df.$$

We then get

$$\omega = \alpha + d\log(\lambda) \wedge p^*(\theta) = \alpha' + d\log(\lambda') \wedge (p')^*(\theta) = \alpha' + (d\log(\lambda) + df) \wedge (p')^*(\theta). \tag{4.12}$$

Taking the restriction to Z then gives

$$\alpha|_Z + (d\log(\lambda))|_Z \wedge \theta = \alpha'|_Z + (d\log(\lambda) + df)|_Z \wedge \theta,$$

hence $\alpha|_Z = \alpha'|_Z + [d\,(f|_Z)] \wedge \theta$. In the particular case where λ and λ' are defined on the same tubular neighborhood $E' = E$, then equation (4.12) implies that $\alpha = \alpha' + df \wedge p^*(\theta)$.

Remark 4.1.17. If M and Z are orientable, we can proceed differently. Lemma 4.1.2 ensures that Z is the zero locus of some defining function f on a tubular neighborhood U. We get a well-defined one-form df/f on $U \setminus Z$, which extends over Z as a b-form. A partition of unity argument shows that any b-form $\omega \in {}^b\Omega^k(M)$ can be written on U as

$$\omega|_U = \alpha \wedge \frac{df}{f} + \beta \tag{4.13}$$

for some $\alpha \in \Omega^{k-1}(U)$ and $\beta \in \Omega^k(U)$ that again have to be interpreted by the rules (4.7).

The decomposition (4.6) again implies that, for a given defining function f, the pullbacks of α and β to Z are uniquely determined. However, α and β themselves are not unique, since they are defined up to summands of the form hdf for $h \in C^\infty(U)$ and $\mu \wedge df$ for $\mu \in \Omega^{k-1}(U)$, respectively. If g is another defining function for Z, then we have $f = gh$ for some $h \in C^\infty(U)$ that is nowhere vanishing. We get

$$\frac{df}{f} = \frac{dg}{g} + d\log(|h|),$$

which implies that

$$\omega|_U = \alpha \wedge \frac{dg}{g} + (\beta + \alpha \wedge d\log(|h|)).$$

Being a Lie algebroid, ${}^bTM$ carries a differential ${}^bd : {}^b\Omega^k(M) \to {}^b\Omega^{k+1}(M)$ that satisfies ${}^bd \circ {}^bd = 0$. This differential is determined by the fact that the restriction ${}^b\Omega^\bullet(M) \to \Omega^\bullet(M \setminus Z)$ is a chain map. So for $\omega \in {}^b\Omega^k(M)$, we have that ${}^bd\omega$ is the unique extension of $d(\omega|_{M\setminus Z})$ over Z as a b-form. In the decomposition (4.8), we have

$${}^bd\big(\alpha + d\log(\lambda) \wedge p^*(\theta)\big) = d\alpha + d\log(\lambda) \wedge d(p^*(\theta)) = d\alpha + d\log(\lambda) \wedge p^*(d\theta).$$

In coordinates $(x_1, \ldots, x_n)$ near $Z \leftrightarrow \{x_1 = 0\}$, we write

$$\omega = \sum_{1<i_2<\cdots<i_k\leq n} f_{i_2,\ldots,i_k} \frac{dx_1}{x_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} + \sum_{1<i_1<\cdots<i_k\leq n} g_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and then

$$\begin{aligned}
{}^bd\omega = &\sum_{1<i_1<\cdots<i_k\leq n} \left(x_1 \frac{\partial g_{i_1,\ldots,i_k}}{\partial x_1}\right) \frac{dx_1}{x_1} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&+ \sum_{1<i_1<\cdots<i_k\leq n} \sum_{j=2}^{n} \frac{\partial g_{i_1,\ldots,i_k}}{\partial x_j} dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \\
&- \sum_{1<i_2<\cdots<i_k\leq n} \sum_{j=2}^{n} \frac{\partial f_{i_2,\ldots,i_k}}{\partial x_j} \frac{dx_1}{x_1} \wedge dx_j \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}.
\end{aligned}$$

As ${}^bd \circ {}^bd = 0$, we can form the b-de Rham complex

$$0 \to {}^b\Omega^0(M) \xrightarrow{{}^bd} {}^b\Omega^1(M) \xrightarrow{{}^bd} {}^b\Omega^2(M) \xrightarrow{{}^bd} \cdots \xrightarrow{{}^bd} 0.$$

Pullbacks and b-derivatives

As expected, one can pull back b-forms under b-maps. Let $f : (X, Z_X) \to (Y, Z_Y)$ be a b-map. We define for all $p \in X$:

$$\left({}^bdf_p\right)^* : {}^bT^*_{f(p)}Y \to {}^bT^*_pX$$

by the rules:

- If $p \notin Z_X$ then $f(p) \notin Z_Y$, hence ${}^bT^*_pX = T^*_pX$ and ${}^bT^*_{f(p)}Y = T^*_{f(p)}Y$. So we can define $\left({}^bdf_p\right)^*$ to be the usual pullback by f.
- On $T^*_{f(p)}Z_Y$ for $p \in Z_X$ we define $\left({}^bdf_p\right)^*$ to be the pullback by $f|_{Z_X}$.

- If y is a local defining function for Z_Y, then we define for $p \in Z_X$:

$$\left({}^b df_p\right)^* \left(\left.\frac{dy}{y}\right|_{f(p)} \right) = \left.\frac{f^*(dy)}{f^*y}\right|_p = \left.\frac{d(f^*y)}{f^*y}\right|_p .$$

Note here that f^*y is indeed a local defining function for Z_X.

We have to check that this definition is consistent. Suppose h is a non-vanishing function, locally defined near Z_Y. Then hy is also locally defining for Z_Y. We have for $p \in Z_X$:

$$\left.\frac{d(hy)}{hy}\right|_{f(p)} = \left.\frac{dh}{h}\right|_{f(p)} + \left.\frac{dy}{y}\right|_{f(p)} \tag{4.14}$$

Now $\left({}^b df_p\right)^*$ takes the left hand side of (4.14) to

$$\left.\frac{d(f^*(hy))}{f^*(hy)}\right|_p ,$$

whereas the right hand side of (4.14) is mapped to

$$f^* \left(\left.\frac{dh}{h}\right|_{f(p)} \right) + \left.\frac{d(f^*y)}{f^*y}\right|_p = \left.\frac{d(f^*h)}{f^*h}\right|_p + \left.\frac{d(f^*y)}{f^*y}\right|_p .$$

Things check out because

$$\left.\frac{d(f^*(hy))}{f^*(hy)}\right|_p = \left.\frac{d(f^*h)}{f^*h}\right|_p + \left.\frac{d(f^*y)}{f^*y}\right|_p .$$

Taking exterior powers, we require $\left({}^b df_p\right)^*$ to distribute over the wedge product, and this results in a well-defined pull-back map $({}^b df)^* : {}^b\Omega^k(Y) \to {}^b\Omega^k(X)$. Indeed, the above rules imply that

$$({}^b df)^* \left(\frac{dy}{y} \wedge \alpha + \beta \right) = \frac{d(f^*y)}{f^*y} \wedge f^*\alpha + f^*\beta, \tag{4.15}$$

where y locally defines Z_Y and α, β are honest de Rham forms. This can be seen as follows: clearly (4.15) holds at points $p \notin Z_X$. At $p \in Z_X$, we have by definition

$$\left({}^b df_p\right)^* \left(\left.\frac{dy}{y}\right|_{f(p)} \right) = \left.\frac{d(f^*y)}{f^*y}\right|_p ,$$

and, since by convention $\alpha_{f(p)} \in \wedge^{k-1} T^*_{f(p)} Z_Y$,

$$\left({}^b df_p\right)^* \left(\alpha_{f(p)}\right) = (f|^*_{Z_X}\alpha)_p .$$

Here $(f|^*_{Z_X}\alpha)_p = (f^*\alpha)_p$ since by convention

$$(f^*\alpha)_p = (i^*(f^*\alpha))_p = \big((f \circ i)^*(\alpha)\big)_p = (f|^*_{Z_X}\alpha)_p,$$

where $i : Z_X \hookrightarrow X$ is the inclusion. Similarly one sees that $\left({}^b df_p\right)^* \left(\beta_{f(p)}\right) = (f^*\beta)_p$, and so Equation (4.15) is correct. Note that the right hand side of (4.15) is indeed a b-form since $f^*y = y \circ f$ is a local defining function for Z_X.

We called this pullback map suggestively $({}^b df)^*$, since in degree one it is the dual map of the b-derivative ${}^b df : {}^bTX \to {}^bTY$. We will only need this b-derivative in the most illuminating case where f is a b-diffeomorphism. We can then push forward vector fields under f, and since $f(Z_X) = Z_Y$, vector fields tangent to Z_X are taken to vector fields tangent to Z_Y. Hence a b-vector field $V \in \Gamma({}^bTX)$ can be pushed forward to $f_*(V) \in \Gamma({}^bTY)$, so that at the level of sections, we can define

$$\Gamma({}^bTX) \to \Gamma({}^bTY) : V \mapsto f_*(V).$$

Since the base map f is a diffeomorphism, this defines a vector bundle map ${}^b df : {}^bTX \to {}^bTY$.

Remark 4.1.18. To keep the notation short, we will also denote the pullback of a b-form ω under a b-map f by $f^*\omega$ instead of $({}^b df)^*(\omega)$.

Some properties in b-calculus

The b-differential ${}^b d$ enjoys the usual properties. It is a degree 1 derivation of the wedge product $\wedge$ (as is the case in the exterior differential algebra for any Lie algebroid), and it commutes with the pullback f^* of a b-map $f : (X, Z_X) \to (Y, Z_Y)$. Indeed, using that the restriction maps

$$r_X : \left({}^b\Omega^\bullet(X), {}^b d\right) \to \left(\Omega^\bullet(X \setminus Z_X), d\right) \qquad \text{and} \qquad r_Y : \left({}^b\Omega^\bullet(Y), {}^b d\right) \to \left(\Omega^\bullet(Y \setminus Z_Y), d\right)$$

are chain maps and that the usual de Rham differential d commutes with pullbacks, we have

$$\begin{aligned} r_X\left(f^*\left({}^b d\omega\right)\right) &= f^*\left(r_Y\left({}^b d\omega\right)\right) = f^*\left(d\left(r_Y(\omega)\right)\right) \\ &= d\left(f^*\left(r_Y(\omega)\right)\right) = d\left(r_X(f^*\omega)\right) = r_X\left({}^b d(f^*\omega)\right). \end{aligned}$$

Hence the equality ${}^b d(f^*\omega) = f^*\left({}^b d\omega\right)$ holds on $X \setminus Z_X$, and extends over Z_X by continuity.

Remark 4.1.19. From now on, we will also denote the b-de Rham differential ${}^b d$ by d.

The usual operations on de Rham differential forms can also be applied to b-forms. For instance, let $\omega \in {}^b\Omega^k(M)$ be a b-form and $X \in {}^b\mathfrak{X}(M)$ a b-vector field. Then the flow $\{\phi_t\}$ of X consists of b-diffeomorphisms, and we can define the Lie derivative of ω in direction of X as

$$\pounds_X\omega = \left.\frac{d}{dt}\right|_{t=0} \phi_t^*\omega,$$

where this pullback is well-defined by the above. Next, it is obvious that we can contract b-forms with a b-vector field. Cartan's magic formula also still holds, i.e. for $\omega \in {}^b\Omega^k(M)$ and $X \in {}^b\mathfrak{X}(M)$, we have

$$\pounds_X\omega = d(\iota_X\omega) + \iota_X d\omega. \tag{4.16}$$

Indeed, using that the restriction map $r : {}^b\Omega^k(M) \to \Omega^k(M \setminus Z)$ commutes with the differentials and that contraction is pointwise, we have

$$r(d(\iota_X\omega)) + r(\iota_X d\omega) = d(r(\iota_X\omega)) + \iota_X(r(d\omega)) = d(\iota_X(r(\omega))) + \iota_X d(r(\omega)) = \pounds_X(r(\omega)).$$

and using linearity of r along with the fact that the ϕ_t are b-maps:

$$\pounds_X(r(\omega)) = \left.\frac{d}{dt}\right|_{t=0} \phi_t^*(r(\omega)) = \left.\frac{d}{dt}\right|_{t=0} r(\phi_t^*\omega) = r\left(\left.\frac{d}{dt}\right|_{t=0} \phi_t^*\omega\right) = r\left(\pounds_X\omega\right).$$

Hence, Cartan's formula (4.16) holds on $M \setminus Z$ and extends over Z by continuity. This recipe applies to many statements about calculus with de Rham forms. For instance, we will also need a version of Lemma 8.2.3 in which the ρ_t are b-diffeomorphisms, the X_t are b-vector fields and the ω_t are b-forms. Lemma 8.2.3 also holds in that b-setup since the b-version of the statement holds on $M \setminus Z$ (use that the restriction r is linear and that the ρ_t are b-maps) and extends over Z by continuity.

4.2 b-Symplectic manifolds

In this section, we introduce the notion of symplectic for the b-category and present b-analogs of the classical Darboux-Moser theorems in symplectic geometry. We also prove the important statement that log-symplectic structures may be regarded as the symplectic structures of the b-category.

4.2.1 Definition and properties

Definition 4.2.1. Let (M, Z) be a $2n$-dimensional b-manifold and $\omega \in {}^b\Omega^2(M)$ a b-form. Then ω is called b-symplectic if it is closed and non-degenerate. Non-degeneracy means that the associated bundle map

$$\omega^\flat : {}^bTM \to {}^bT^*M : X \mapsto \iota_X\omega$$

is an isomorphism, or equivalently, that ω^n is nowhere vanishing as an element of ${}^b\Omega^{2n}(M)$.

Example 4.2.2. We take the b-manifold (M, Z) where $M = (\mathbb{R}^{2n}, x_1, y_1, \ldots, x_n, y_n)$ and Z is the hyperplane $y_1 = 0$. The b-form

$$\omega = dx_1 \wedge \frac{dy_1}{y_1} + \sum_{i=2}^{n} dx_i \wedge dy_i$$

is closed, since its restriction to $M \setminus Z$ is a closed de Rham form. And ω is non-degenerate, since

$$\omega^n = n! dx_1 \wedge \frac{dy_1}{y_1} \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dx_n \wedge dy_n$$

is a nowhere vanishing b-form. Hence ω is a b-symplectic form on (M, Z). Note that the b-bivector field $\Pi \in \Gamma\big(\wedge^2 ({}^bTM)\big)$ dual to ω is of the form (3.2), whence applying the anchor map ρ to it yields a log-symplectic structure. This is no coincidence; we will show later that the duals of b-symplectic forms are log-symplectic.

That this example is the local prototype of all b-symplectic manifolds is the content of the b-Darboux theorem, which we will prove soon. But first, we show that b-symplectic structures are closely related to cosymplectic structures.

Definition 4.2.3. A cosymplectic stucture on a manifold M^{2n+1} is a pair of differential forms (α, ω), where $\alpha \in \Omega^1(M)$ and $\omega \in \Omega^2(M)$ are closed, such that $\alpha \wedge \omega^n$ is a volume form.

Remark 4.2.4. We have the equivalence

$$(\alpha \wedge \omega^n)_p \neq 0 \Leftrightarrow \begin{cases} \alpha_p \neq 0 \\ \omega_p : \operatorname{Ker}(\alpha_p) \times \operatorname{Ker}(\alpha_p) \to \mathbb{R} \text{ is non-degenerate} \end{cases} . \tag{4.17}$$

Indeed, first assume that the right hand side of (4.17) holds. Then $\dim(\operatorname{Ker}(\alpha_p)) = 2n$, and ω_p^n is nonzero on $\operatorname{Ker}(\alpha_p)$. So there exist $v_2, \ldots, v_{2n+1} \in \operatorname{Ker}(\alpha_p)$ such that $\omega_p^n(v_2, \ldots, v_{2n+1}) \neq 0$. Let $v_1 \in T_pM \setminus \operatorname{Ker}(\alpha_p)$. Then

$$(\alpha \wedge \omega^n)_p(v_1, v_2 \ldots, v_{2n+1}) = \alpha_p(v_1)\omega_p^n(v_2, \ldots, v_{2n+1}) \neq 0.$$

Conversely, if $(\alpha \wedge \omega^n)_p \neq 0$, then in particular $\alpha_p \neq 0$ hence $\dim(\operatorname{Ker}(\alpha_p)) = 2n$. So we can choose a basis $\{v_1, \ldots, v_{2n+1}\}$ of T_pM where $v_2, \ldots, v_{2n+1} \in \operatorname{Ker}(\alpha_p)$ and $v_1 \notin \operatorname{Ker}(\alpha_p)$. Since $(\alpha \wedge \omega^n)_p \neq 0$, it evaluates every basis of T_pM to a non-zero number. So

$$(\alpha \wedge \omega^n)_p(v_1, v_2, \ldots, v_{2n+1}) = \alpha_p(v_1)\omega_p^n(v_2, \ldots, v_{2n+1}) \neq 0,$$

which implies that $\omega_p^n(v_2, \ldots, v_{2n+1}) \neq 0$. Hence ω_p^n is nonzero on $\operatorname{Ker}(\alpha_p)$, which shows that the right hand side of (4.17) holds.

Lemma 4.2.5. *Let (M^{2n}, Z) be a b-manifold with b-symplectic form ω. By Lemma 4.1.16, we can decompose*

$$\omega = \alpha + d\log(\lambda) \wedge p^*(\theta),$$

for some choice of distance function λ. Here $\theta \in \Omega^1(Z)$, $\alpha \in \Omega^2(M)$ and $p : E \to Z$ is the projection in a tubular neighborhood of Z. Denote by $i : Z \hookrightarrow M$ the inclusion. Then:

i) The pair $(\theta, i^*\alpha)$ is a cosymplectic structure on Z.

ii) The codimension-one foliation of Z defined by θ is intrinsically defined. For each leaf $L \overset{i_L}{\hookrightarrow} Z$ of this foliation, the form $i_L^*(i^*\alpha)$ is an intrinsically defined symplectic form on L.

Proof. i) We have $0 = d\omega = d\alpha + d\log(\lambda) \wedge p^*(d\theta)$. The discussion following Lemma 4.1.16 shows that $d\alpha$ and $d\theta$ are uniquely determined (for fixed λ). Hence they must be zero: $d\theta = 0$ and $d\alpha = 0$. Then also $d(i^*\alpha) = i^*(d\alpha) = 0$. It remains to show that $\theta \wedge (i^*\alpha)^{n-1}$ is a volume form on Z. Non-degeneracy of ω implies that

$$\omega^n = \alpha^n + n\alpha^{n-1} \wedge d\log(\lambda) \wedge p^*(\theta)$$

is a nowhere vanishing b-form. In particular, it is non-vanishing on Z. Hence by the conventions established:

$$\omega^n|_Z = (i^*\alpha)^n + n(i^*\alpha)^{n-1} \wedge d\log(\lambda)|_Z \wedge \theta$$

is nowhere vanishing. Now $(i^*\alpha)^n = 0$ since it is a $2n$-form on the $(2n-1)$-dimensional manifold Z. Hence

$$(i^*\alpha)^{n-1} \wedge d\log(\lambda)|_Z \wedge \theta$$

is nowhere vanishing. In particular, $(i^*\alpha)^{n-1} \wedge \theta$ does not vanish on Z.

ii) The discussion following Lemma 4.1.16 shows that θ does not depend on the choice of distance function λ, i.e. it comes canonically with ω. The previous point *i)* implies in particular that θ is nowhere zero. Hence it follows that θ gives a codimension-one foliation of Z that is intrinsically defined. Next, the discussion following Lemma 4.1.16 shows that $i^*\alpha = \alpha|_Z$ is intrinsically defined modulo summands of the form $[d(f|_Z)] \wedge \theta$ for some $f \in C^\infty(M)$. If L is a leaf that integrates the foliation $\mathrm{Ker}(\theta)$, then

$$i_L^*\left([d(f|_Z)] \wedge \theta\right) = i_L^*\left([d(f|_Z)]\right) \wedge i_L^*(\theta) = 0.$$

So $i_L^*(i^*\alpha)$ is intrinsically defined. Clearly $i_L^*(i^*\alpha)$ is closed since α is closed, and (4.17) shows that $i_L^*(i^*\alpha)$ is also non-degenerate. Hence $i_L^*(i^*\alpha)$ is a symplectic form on L.

□

We already hinted at the fact that the dual bivector Π of a b-symplectic form ω on (M, Z) is log-symplectic. One might expect that the codimension-one symplectic foliation of Z which comes canonically with ω coincides with the foliation of Z induced by $\Pi|_Z$. We will show later that this is indeed the case.

Remark 4.2.6. We can derive a particularly neat coordinate expression for the restriction $\omega|_Z$ of a b-symplectic form ω on (M, Z). What follows is an expanded version of [GMP2, Remark 13]. Choose $p \in Z$ and let f be a local defining function for Z. We write near p:

$$\omega = \alpha \wedge \frac{df}{f} + \beta.$$

Let $i : Z \hookrightarrow M$ be the inclusion map, and denote $\tilde{\alpha} := i^*(\alpha)$ and $\tilde{\beta} = i^*(\beta)$. We have

$$0 = d\omega = d\alpha \wedge \frac{df}{f} + d\beta,$$

and since $d\tilde{\alpha}$ and $d\tilde{\beta}$ are uniquely determined (as argued in Remark 4.1.17), they have to be zero. So $\tilde{\alpha}$ and $\tilde{\beta}$ are closed. The same proof as that of Lemma 4.2.5 shows that $\tilde{\alpha} \wedge \tilde{\beta}^{n-1}$ is nowhere vanishing. Consequently, we have a closed form $\tilde{\beta} \in \Omega^2(Z)$ for which $\tilde{\beta}^n = 0$ (being a $2n$-form on a $(2n-1)$-manifold) and $\tilde{\beta}^{n-1}$ is nowhere zero (since the same holds for $\tilde{\alpha} \wedge \tilde{\beta}^{n-1}$). By the presymplectic Darboux theorem 1.3.16, we find coordinates $(x_1, x_2, y_2, \dots, x_n, y_n)$ on Z around p such that

$$\tilde{\beta} = \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Since $\tilde{\alpha}$ is closed, it is locally exact by the Poincaré Lemma. Hence there exists a smooth function g defined near p on Z so that

$$\tilde{\alpha} = dg = \frac{\partial g}{\partial x_1} dx_1 + \frac{\partial g}{\partial x_2} dx_2 + \frac{\partial g}{\partial y_2} dy_2 + \cdots + \frac{\partial g}{\partial x_n} dx_n + \frac{\partial g}{\partial y_n} dy_n.$$

We have that

$$\tilde{\alpha} \wedge \tilde{\beta}^{n-1} = (n-1)! \frac{\partial g}{\partial x_1} dx_1 \wedge dx_2 \wedge dy_2 \wedge \cdots \wedge dx_n \wedge dy_n$$

is non-vanishing, hence $\partial g / \partial x_1$ is non-vanishing. It follows that the map

$$(x_1, x_2, y_2, \dots, x_n, y_n) \mapsto (g, x_2, y_2, \dots, x_n, y_n)$$

is a change of coordinates on Z near p, since its Jacobian determinant is given by

$$\det \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \cdots & \frac{\partial g}{\partial y_n} \\ \frac{\partial x_2}{\partial x_1} & \frac{\partial x_2}{\partial x_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial y_n} \end{bmatrix} = \det \left[\begin{array}{c|c} \frac{\partial g}{\partial x_1} & * \\ \hline 0 & I_{(2n-2)\times(2n-2)} \end{array} \right] = \frac{\partial g}{\partial x_1},$$

which is non-vanishing. We may hence assume that $\tilde{\alpha} = dx_1$. Now let $\pi : E \to Z$ be a tubular neighborhood of Z and consider a local trivialization $U \times \mathbb{R}$ near $p \in U$. Shrinking U if necessary, we can assume that $(x_1, x_2, y_2, \dots, x_n, y_n)$ are coordinates on U, and we let t be the coordinate on $\mathbb{R}$. Then $(\pi^* x_1, \pi^* x_2, \pi^* y_2, \dots, \pi^* x_n, \pi^* y_n, t)$ are coordinates on $U \times \mathbb{R}$, which we just still denote by $(x_1, x_2, y_2, \dots, x_n, y_n, t)$ as is usual. Since f is a local defining function for $Z \leftrightarrow \{t = 0\}$, we have $f = ht$ for some h non-vanishing. It follows that the map

$$(x_1, x_2, y_2, \dots, x_n, y_n, t) \mapsto (x_1, x_2, y_2, \dots, x_n, y_n, f)$$

is a change of coordinates on M near p, since its Jacobian determinant at $p \in Z$ is

$$\frac{\partial f}{\partial t}(p) = \left(\frac{\partial h}{\partial t} t + h \right)(p) = h(p),$$

which is nonzero. Hence in the coordinates $(x_1, x_2, y_2 \dots, x_n, y_n, f)$ on M near p, we have

$$\omega|_Z = \tilde{\alpha} \wedge \frac{df}{f}\bigg|_Z + \tilde{\beta} = \left(dx_1 \wedge \frac{df}{f} + \sum_{i=2}^{n} dx_i \wedge dy_i \right)\bigg|_Z.$$

Renaming $y_1 := f$, we obtain the local expression we were looking for:

$$\omega|_Z = \left(dx_1 \wedge \frac{dy_1}{y_1} + \sum_{i=2}^{n} dx_i \wedge dy_i \right)\Bigg|_Z . \tag{4.18}$$

We need one more ingredient to prove the b-Darboux theorem, namely a b-version of the local Moser theorem.

Theorem 4.2.7 (Local b-Moser)**.** *Let ω_0 and ω_1 be two b-symplectic forms on (M, Z). If $\omega_0|_Z = \omega_1|_Z$, then there exist neighborhoods U_0, U_1 of Z in M and a diffeomorphism $\gamma : U_0 \to U_1$ such that $\gamma|_Z = Id_Z$ and $\gamma^*\omega_1 = \omega_0$.*

Proof. Define $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$ for $t \in [0, 1]$. We will prove that there exists a neighborhood U of Z in M and an isotopy $\gamma_t : U \to M$, such that $\gamma_t|_Z = \mathrm{Id}_Z$ and $\gamma_t^*\omega_t = \omega_0$ for all $t \in [0, 1]$. This then gives the desired diffeomorphism $\gamma_1 : U \to \gamma_1(U)$ between opens around Z, with $\gamma_1|_Z = \mathrm{Id}_Z$ and $\gamma_1^*\omega_1 = \omega_0$. Suppose $\{\gamma_t\}_{t\in[0,1]}$ is an isotopy such that $\gamma_t|_Z = \mathrm{Id}_Z$ for all $t \in [0, 1]$. If $\{v_t\}_{t\in[0,1]}$ is the associated time dependent vector field, defined by

$$v_t = \frac{d\gamma_t}{dt} \circ \gamma_t^{-1},$$

then v_t is a b-vector field vanishing on Z. Indeed, for $p \in Z$ we have

$$v_t(p) = \frac{d\gamma_t}{dt}\big(\gamma_t^{-1}(p)\big) = \frac{d}{ds}\bigg|_{s=t} \gamma_s(\gamma_t^{-1}(p)) = \frac{d}{ds}\bigg|_{s=t} \gamma_s(p) = \frac{d}{ds}\bigg|_{s=t} p = 0,$$

showing that v_t vanishes on Z (in particular, it is tangent to Z). As in Theorem 1.3.10, we have the following equivalences for such an isotopy:

$$\begin{aligned}
\gamma_t^*\omega_t = \omega_0 \ \forall t \in [0,1] &\Leftrightarrow \frac{d}{dt}(\gamma_t^*\omega_t) = 0 \\
&\Leftrightarrow \gamma_t^*\left(\pounds_{v_t}\omega_t + \frac{d}{dt}\omega_t \right) = 0 \\
&\Leftrightarrow \pounds_{v_t}\omega_t = \omega_0 - \omega_1 \\
&\Leftrightarrow d\left(\iota_{v_t}\omega_t\right) = \omega_0 - \omega_1.
\end{aligned} \tag{4.19}$$

Above manipulations are allowed, as is noted in the paragraph following Remark 4.1.19. Because $(\omega_0 - \omega_1)|_Z = 0$, the b-form $\omega_0 - \omega_1$ is an honest de Rham form by Lemma 4.1.15. Since it is closed, the relative Poincaré Lemma 1.3.9 gives a neighborhood V of Z on which $\omega_0 - \omega_1 = d\beta$ for some $\beta \in \Omega^1(V)$ with $\beta|_Z = 0$. Hence to solve (4.19), it is sufficient to solve

$$\iota_{v_t}\omega_t = \beta \tag{4.20}$$

for v_t. Note that $\omega_t(p) = \omega_0(p)$ for $p \in Z$, so that each ω_t is non-degenerate on Z. Since non-degeneracy is an open condition, each ω_t is non-degenerate on some open neighborhood V_t of Z. Using the Tube Lemma 1.3.13 and shrinking V if necessary, we can assume that all ω_t for $t \in [0, 1]$ are b-symplectic on V. We can now solve (4.20) on V as

$$v_t = (\omega_t^\flat)^{-1}(\beta).$$

The same argument as in the proof of Theorem 1.3.14 gives an open U around Z so that the isotopy $\{\gamma_t\}$ integrating $\{v_t\}$ is defined on U for all $t \in [0, 1]$:

$$\gamma : [0, 1] \times U \to M.$$

Since the v_t vanish on Z, we have $\gamma_t|_Z = \mathrm{Id}_Z$ for all $t \in [0, 1]$. This finishes the proof. □

As in the symplectic case, the local b-Moser theorem can be used to prove a local normal form result for b-symplectic forms, an analogue of the classical Darboux theorem.

Theorem 4.2.8 (b-Darboux)**.** *Let ω be a b-symplectic form on (M^{2n}, Z) and $p \in Z$. Then we can find a coordinate chart $(U, x_1, y_1, \ldots, x_n, y_n)$ around p such that on U, the hypersurface Z is locally defined by $y_1 = 0$ and*

$$\omega|_U = dx_1 \wedge \frac{dy_1}{y_1} + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Proof. By Remark 4.2.6, we find coordinates $(V, x_1', y_1', \ldots, x_n', y_n')$ around p such that Z is locally defined by $y_1' = 0$ and

$$\omega|_{V\cap Z} = \left(dx_1' \wedge \frac{dy_1'}{y_1'} + \sum_{i=2}^{n} dx_i' \wedge dy_i' \right)\Bigg|_Z .$$

The local b-Moser theorem gives neighborhoods U_0 and U_1 of $V \cap Z$ inside V and a diffeomorphism $\gamma : U_0 \to U_1$ such that $\gamma|_{V\cap Z} = \mathrm{Id}_{V\cap Z}$ and

$$\gamma^* \left(dx_1' \wedge \frac{dy_1'}{y_1'} + \sum_{i=2}^{n} dx_i' \wedge dy_i' \right) = d(x_1' \circ \gamma) \wedge \frac{d(y_1' \circ \gamma)}{y_1' \circ \gamma} + \sum_{i=2}^{n} d(x_i' \circ \gamma) \wedge d(y_i' \circ \gamma) = \omega|_{U_0}.$$

So we only have to set $U := U_0$ and define new coordinates

$$(x_1, y_1, \ldots, x_n, y_n) := (x_1' \circ \gamma, y_1' \circ \gamma, \ldots, x_n' \circ \gamma, y_n' \circ \gamma).$$

□

4.2.2 Log-symplectic equals b-symplectic

We will now show that a log-symplectic structure (Π, M, Z) can be regarded as a b-symplectic structure on (M, Z), and vice versa. Recall that we identify the sections of ${}^bTM$ with the set of vector field on M that are tangent to Z, as follows. The anchor map $\rho : {}^bTM \to TM$ induces a $C^\infty(M)$-linear map on sections

$$\widetilde{\rho} : \Gamma({}^bTM) \to \Gamma(TM) : X \mapsto \rho \circ X$$

that is injective since ρ is the identity map on the dense subset $M \setminus Z$. As ρ restricts over Z to a bundle epimorphism $\rho|_Z : {}^bTM|_Z \to TZ$, the map $\widetilde{\rho}$ is a $C^\infty(M)$-isomorphism onto the submodule of vector fields tangent to Z:

$$\widetilde{\rho} : \Gamma({}^bTM) \xrightarrow{\sim} \{Y \in \mathfrak{X}(M) : Y(p) \in T_pZ \text{ for all } p \in Z\}.$$

Taking exterior powers, also

$$\widetilde{\rho} : \Gamma(\wedge^2({}^bTM)) \xrightarrow{\sim} \{Y \in \mathfrak{X}^2(M) : Y(p) \in \wedge^2 T_pZ \text{ for all } p \in Z\}. \tag{4.21}$$

Under this correspondence, we have the following:

Lemma 4.2.9. *A log-symplectic structure Π on M^{2n} with singular locus Z is the same thing as a non-degenerate section $\Pi \in \Gamma(\wedge^2({}^bTM))$ satisfying $[\Pi, \Pi] = 0$, where ${}^bTM$ is the b-tangent bundle of (M, Z).*

Proof. First assume Π is a log-symplectic structure on M^{2n} with singular locus Z. We have showed in Corollary 3.2.3 that Π is then tangent to Z, i.e. $\Pi_p \in \wedge^2 T_pZ$ for all $p \in Z$. This means that Π can naturally be considered as a b-bivector $\Pi \in \Gamma(\wedge^2({}^bTM))$, applying $(\widetilde{\rho})^{-1}$ as above. To show that $\Pi \in \Gamma(\wedge^2({}^bTM))$ is non-degenerate, we only need to look at points in Z, since at $p \notin Z$ we have that $\Pi_p^n \in \wedge^{2n}T_pM = \wedge^{2n}({}^bT_pM)$ is nonzero as Π is log-symplectic. But around $p \in Z$, we have by Theorem 3.2.2 that

$$\Pi = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i},$$

where Z is locally given by $y_1 = 0$. Hence

$$\Pi^n = n! y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} \wedge \frac{\partial}{\partial y_n} \in \Gamma(\wedge^{2n}TM),$$

which is non-vanishing as a section of $\wedge^{2n}({}^bTM)$. Conversely, assume (M^{2n}, Z) is a b-manifold and we are given a non-degenerate section $\Pi \in \Gamma(\wedge^2({}^bTM)) \cong {}^b\mathfrak{X}^2(M) \subset \mathfrak{X}^2(M)$ such that $[\Pi, \Pi] = 0$. Again, we only have to check the log-symplectic condition near Z. Let $p \in Z$ and choose adapted coordinates $(x_1, y_1, \ldots, x_n, y_n)$ near p such that locally Z is given by $x_1 = 0$. We can then write

$$\Pi^n = f x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} \wedge \cdots \wedge \frac{\partial}{\partial x_n} \wedge \frac{\partial}{\partial y_n} \in \Gamma(\wedge^{2n}({}^bTM)) \tag{4.22}$$

for some smooth function f defined near p. Since $\Pi \in \Gamma(\wedge^2({}^bTM))$ is non-degenerate, we have that (4.22) is non-vanishing as a b-$2n$-vector field. Hence f is non-vanishing, and this implies that Π^n vanishes linearly on Z when considered as a section of $\wedge^{2n}TM$. □

So log-symplectic is the notion of non-degenerate Poisson in the b-category. This shows why the b-category is useful for our purposes: by considering a log-symplectic structure as a b-bivector, we get rid of its singularities. As one might expect, it is also true in the b-category that non-degenerate Poisson and symplectic are equivalent notions. This then establishes the following correspondence between log-symplectic and b-symplectic structures:

Theorem 4.2.10. *Given a b-manifold (M^{2n}, Z), a b-form $\omega \in {}^b\Omega^2(M)$ is b-symplectic if and only if its dual bivector Π is a log-symplectic structure on M with singular locus Z.*

Proof. Given a b-symplectic form $\omega \in {}^b\Omega^2(M)$, we have that $\omega^\flat : {}^bTM \to {}^bT^*M$ can be inverted to define a b-bivector $\Pi^\sharp = -(\omega^\flat)^{-1} : {}^bT^*M \to {}^bTM$. Applying the anchor map $\widetilde{\rho} : \Gamma(\wedge^2({}^bTM)) \to \Gamma(\wedge^2 TM)$ gives a bivector $\widetilde{\rho}(\Pi)$, which we call the dual bivector of ω. Now, checking that $\widetilde{\rho}(\Pi)$ is log-symplectic only needs to be done locally near Z (After all, away from Z we have that ω is an honest symplectic de Rham form and that $\widetilde{\rho}$ is the identity map, hence the dual bivector field $\Pi = \widetilde{\rho}(\Pi)$ non-degenerate Poisson over there). By the b-Darboux theorem, we can write near $p \in Z \leftrightarrow \{y_1 = 0\}$:

$$\omega = dx_1 \wedge \frac{dy_1}{y_1} + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

In these coordinates,

$$\Pi = -\omega^{-1} = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} \in \Gamma(\wedge^2({}^bTM)),$$

hence

$$\widetilde{\rho}(\Pi) = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} \in \Gamma(\wedge^2 TM),$$

which is clearly a log-symplectic structure with singular locus $Z \leftrightarrow \{y_1 = 0\}$. Conversely, assume $\Pi \in \Gamma(\wedge^2 TM)$ is log-symplectic with singular locus Z. Then Π is tangent to Z, and previous lemma shows that $(\widetilde{\rho})^{-1}(\Pi)$ is a non-degenerate b-bivector on (M, Z). So we can invert $\left((\widetilde{\rho})^{-1}(\Pi)\right)^\sharp$ and obtain a b-form $\omega \in {}^b\Omega^2(M)$ by $\omega^\flat := -\left(\left((\widetilde{\rho}^{-1})(\Pi)\right)^\sharp\right)^{-1} : {}^bTM \to {}^bT^*M$. We show that this is a b-symplectic form. Away from Z, there is again nothing to prove, since there $\omega^\flat = -\left(\Pi^\sharp\right)^{-1}$ is plain symplectic. Around $p \in Z \leftrightarrow \{y_1 = 0\}$, we can write

$$\Pi = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} \in \Gamma(\wedge^2 TM)$$

hence

$$(\widetilde{\rho})^{-1}(\Pi) = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} \in \Gamma(\wedge^2({}^bTM))$$

and so

$$\omega = -((\widetilde{\rho})^{-1}(\Pi))^{-1} = dx_1 \wedge \frac{dy_1}{y_1} + \sum_{i=2}^{n} dx_i \wedge dy_i,$$

which is clearly a b-symplectic form. □

So this dual approach allows us to regard log-symplectic structures as the symplectic structures on the b-tangent bundle, and in this way symplectic techniques can be used in the study of log-symplectic structures. In what follows, we will move back and forth between log-symplectic stuctures and their associated b-symplectic structures, depending on which point of view is the most convenient. It is important to keep in mind that both notions are equivalent. We will also denote by Π the dual bivector $\widetilde{\rho}(\Pi)$ of a b-symplectic form ω, to keep the notation concise.

Remark 4.2.11. We saw in Lemma 4.2.5 that with a b-symplectic form $\omega \in {}^b\Omega^2(M)$ on (M, Z) comes an intrinsically defined codimension one symplectic foliation of Z. It is no longer a mystery what this foliation is. Since the dual bivector $\Pi = -\omega^{-1} \in \Gamma(\wedge^2 TM)$ is log-symplectic with singular locus Z, we know that $\Pi|_Z$ induces a codimension one symplectic foliation on Z. It is only natural to expect that these two foliations coincide. This is readily checked. Let $\omega = \alpha + d\log(\lambda) \wedge p^*(\theta)$ for some choice of distance function λ, as in Lemma 4.2.5. The leaves of the foliation induced by ω integrate $\mathrm{Ker}(\theta)$, and each such leaf L is endowed with a symplectic form $i_L^*\tilde{\alpha}$, where $\tilde{\alpha}$ is the pullback of α to Z. To show that this foliation is indeed the foliation of $\Pi|_Z$, we have to show that

$$\begin{cases} \mathrm{Ker}(\theta_p) = \mathrm{Im}(\Pi_p^\sharp) \text{ at } p \in Z \\ i_L^*\tilde{\alpha} = -\Pi|_L^{-1} \text{ for each leaf } L \end{cases}. \tag{4.23}$$

Since θ and $i_L^*\tilde{\alpha}$ are intrinsic and both statements in (4.23) are pointwise, we can check them in coordinates choosing any distance function λ. By the b-Darboux theorem, we can choose coordinates $(x_1, y_1, \ldots, x_n, y_n)$ near $p \in Z \leftrightarrow \{y_1 = 0\}$ such that

$$\omega|_Z = dx_1 \wedge \frac{dy_1}{y_1} + \sum_{i=2}^{n} dx_i \wedge dy_i = dx_1 \wedge d\log|y_1| + \sum_{i=2}^{n} dx_i \wedge dy_i,$$

where $|y_1|$ is a local distance function. The dual bivector $\Pi = -\omega^{-1}$ then satisfies

$$\begin{aligned}\Pi|_Z &= \frac{\partial}{\partial x_1} \wedge y_1 \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} \in \Gamma(\wedge^2({}^bTM)) \\ &= \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} \in \Gamma(\wedge^2 TM).\end{aligned}$$

So we see that $\theta = -dx_1$, hence at $p \in Z$:

$$\operatorname{Ker}(\theta_p) = \operatorname{span}\left\{ \left.\frac{\partial}{\partial x_2}\right|_p, \left.\frac{\partial}{\partial y_2}\right|_p, \dots, \left.\frac{\partial}{\partial x_n}\right|_p, \left.\frac{\partial}{\partial y_n}\right|_p \right\} = \operatorname{Im}(\Pi_p^\sharp).$$

That is, the leaves of both foliations are the level sets of x_1. On such a leaf L, we have

$$i_L^* \tilde{\alpha} = \sum_{i=2}^{n} dx_i \wedge dy_i = -\left.\left(\sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} \right)\right|_L^{-1} = -\Pi|_L^{-1}.$$

So if $\Pi \in \Gamma(\wedge^2 TM)$ is log-symplectic, then we have established that the symplectic foliation of $\Pi|_Z$ has a closed defining one-form (namely θ above) and a closed two-form that pulls back to the symplectic form on each leaf (namely $\tilde{\alpha}$ above). This is a rather special property, which will play an important role in next chapter.

Example 4.2.12. The real affine group $A(n)$ is the group of affine transformations $x \mapsto Ax + a$ in $\mathbb{R}^n$. Thus, the affine group is parameterized by pairs (A, a) consisting of an invertible matrix $A \in GL_n(\mathbb{R})$ and a vector $a \in \mathbb{R}^n$. This correspondence can be used to give $A(n)$ the structure of a smooth manifold of dimension $\dim(GL_n(\mathbb{R}) \times \mathbb{R}^n) = n(n+1)$. As a group however, $A(n)$ is not the Cartesian product of the groups $GL_n(\mathbb{R})$ and $\mathbb{R}^n$ since the group multiplication law is

$$(A, a) \cdot (B, b) = (AB, a + Ab).$$

So in fact, $A(n)$ as a group is the semi-direct product $A(n) = GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$, and this multiplication law gives $A(n)$ a Lie group structure.

Let us consider the 2-dimensional affine group $A(1)$ consisting of transformations in $\mathbb{R}$ of the form $x \mapsto ax + b$, where $a \in \mathbb{R}_0$ and $b \in \mathbb{R}$. The group multiplication law is $(a, b) \cdot (c, d) = (ac, ad + b)$, with neutral element $e = (1, 0)$. The Lie algebra $\mathfrak{a}(1)$ of $A(1)$ is isomorphic to the vector space of left invariant vector fields, via $\mathfrak{a}(1) \ni v \mapsto v^L$. Here $v^L(p) = (dL_p)_e(v)$, where L_p is left multiplication by $p \in A(1)$. In the coordinates (a, b) on $A(1)$, we have correspondingly a basis

$$\left\{ \left.\frac{\partial}{\partial a}\right|_e, \left.\frac{\partial}{\partial b}\right|_e \right\}$$

of $T_e A_1 = \mathfrak{a}(1)$. Hence, a basis for the left invariant vector fields is $\{v_1, v_2\}$, where

$$\begin{aligned}v_1 = dL_{(a,b)}\left(\left.\frac{\partial}{\partial a}\right|_{(1,0)} \right) &= \left.\frac{d}{dt}\right|_{t=0} L_{(a,b)}\big((1,0) + t(1,0)\big) = \left.\frac{d}{dt}\right|_{t=0} L_{(a,b)}(1+t, 0) \\ &= \left.\frac{d}{dt}\right|_{t=0} (a + at, b) = (a, 0) = a\frac{\partial}{\partial a}\end{aligned}$$

and

$$v_2 = dL_{(a,b)}\left(\left.\frac{\partial}{\partial b}\right|_{(1,0)} \right) = \left.\frac{d}{dt}\right|_{t=0} L_{(a,b)}\big((1,0) + t(0,1)\big) = \left.\frac{d}{dt}\right|_{t=0} L_{(a,b)}(1, t)$$

$$= \left.\frac{d}{dt}\right|_{t=0} (a, at + b) = (0, a) = a\frac{\partial}{\partial b}.$$

Now note that the Lie bracket of these vector fields is

$$\left[a\frac{\partial}{\partial a}, a\frac{\partial}{\partial b}\right] = a\frac{\partial}{\partial a}\left(a\frac{\partial}{\partial b}\right) - a\frac{\partial}{\partial b}\left(a\frac{\partial}{\partial a}\right) = a\left(\frac{\partial}{\partial b} + a\frac{\partial^2}{\partial a \partial b}\right) - a^2\frac{\partial^2}{\partial b \partial a} = a\frac{\partial}{\partial b}.$$

Hence, the Lie algebra $\mathfrak{a}(1)$ is span$\{v_1, v_2\}$ with $[v_1, v_2] = v_2$. This gives a corresponding Lie-Poisson structure Π on the dual $\mathfrak{a}(1)^*$. If (μ_1, μ_2) are the coordinates on $\mathfrak{a}(1)^*$ induced by the dual basis $\{v_1^*, v_2^*\}$, then Example 2.4.10 shows that

$$\Pi = \{\mu_1, \mu_2\}\frac{\partial}{\partial \mu_1} \wedge \frac{\partial}{\partial \mu_2} = \mu_2\frac{\partial}{\partial \mu_1} \wedge \frac{\partial}{\partial \mu_2}.$$

So the Lie-Poisson structure Π is log-symplectic, with critical locus the axis $\{\mu_2 = 0\}$ and dual b-symplectic form

$$\omega = d\mu_1 \wedge \frac{d\mu_2}{\mu_2}.$$

The symplectic foliation of $\mathfrak{a}(1)^*$ induced by Π integrates $\mathrm{Im}(\Pi^\sharp)$ and we have

$$\mathrm{Im}\left(\Pi^\sharp_{(\mu_1,\mu_2)}\right) = \mathrm{span}\left\{\Pi^\sharp(d\mu_1), \Pi^\sharp(d\mu_2)\right\} = \mathrm{span}\left\{\mu_2\frac{\partial}{\partial \mu_2}, -\mu_2\frac{\partial}{\partial \mu_1}\right\}.$$

Hence, the exceptional hypersurface $\{\mu_2 = 0\}$ is the union of symplectic leaves of dimension 0 (i.e. all points on the line), and the open upper and lower half-planes are symplectic leaves of dimension 2. This is consistent with previous remark: the foliation of $\{\mu_2 = 0\}$ is defined by the closed one-form $d\mu_1$ on $\{\mu_2 = 0\}$ and the zero form is a closed two-form on $\{\mu_2 = 0\}$ which pulls back to the symplectic form on each leaf.

An interesting side remark is that the Lie algebra $\mathfrak{a}(1)$ considered in the above example is the only non-abelian two-dimensional Lie algebra, up to isomorphism.

4.2.3 Modular vector fields of b-symplectic manifolds

In Section 2.9, we introduced modular vector fields on Poisson manifolds. Recall the definition:

Definition 4.2.13. Let (M, Π) be an orientable Poisson manifold and Ω a volume form on it. Denote by X_f the Hamiltonian vector field associated to the smooth function $f \in C^\infty(M)$. The modular vector field X_Π^Ω is the derivation given by the mapping

$$X_\Pi^\Omega : C^\infty(M) \to C^\infty(M) : f \mapsto \frac{\mathcal{L}_{X_f}\Omega}{\Omega}.$$

Recall that the modular vector field X_Π^Ω is a Poisson vector field. When Π is log-symplectic, then X_Π^Ω enjoys some convenient additional properties.

Lemma 4.2.14. *Let (M^{2n}, Z) be a b-symplectic manifold, with b-symplectic form ω and dual log-symplectic bivector Π. The b-Darboux theorem gives coordinates $(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, z, t)$ around $p \in Z$ so that ω can be written as*

$$\omega = \sum_{i=1}^{n-1} dx_i \wedge dy_i + \frac{1}{z}dz \wedge dt.$$

Consider the locally defined volume form

$$\Omega = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_{n-1} \wedge dy_{n-1} \wedge dz \wedge dt.$$

Working in these local coordinates, the modular vector field X_Π^Ω *associated to* Π *and* Ω *is given by*

$$X_\Pi^\Omega = -\frac{\partial}{\partial t}.$$

Proof. Since

$$\Pi = \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + z\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial t},$$

Lemma 2.7.4 implies that the Hamiltonian vector field of $f \in C^\infty(M)$ is

$$X_f = \sum_{i=1}^{n-1} \left(\frac{\partial f}{\partial x_i}\frac{\partial}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial}{\partial x_i} \right) + z\frac{\partial f}{\partial z}\frac{\partial}{\partial t} - z\frac{\partial f}{\partial t}\frac{\partial}{\partial z}.$$

Using Cartan's magic formula, we have

$$\begin{aligned} \pounds_{X_f}\Omega = \sum_{i=1}^{n-1} \Bigg[& \frac{\partial f}{\partial x_i} \pounds_{\frac{\partial}{\partial y_i}}(dx_1 \wedge \cdots \wedge dt) + d\left(\frac{\partial f}{\partial x_i}\right) \wedge \iota_{\frac{\partial}{\partial y_i}}(dx_1 \wedge \cdots \wedge dt) \\ & - \frac{\partial f}{\partial y_i} \pounds_{\frac{\partial}{\partial x_i}}(dx_1 \wedge \cdots \wedge dt) - d\left(\frac{\partial f}{\partial y_i}\right) \wedge \iota_{\frac{\partial}{\partial x_i}}(dx_1 \wedge \cdots \wedge dt) \Bigg] \\ & + z\frac{\partial f}{\partial z} \pounds_{\frac{\partial}{\partial t}}(dx_1 \wedge \cdots \wedge dt) + d\left(z\frac{\partial f}{\partial z}\right) \wedge \iota_{\frac{\partial}{\partial t}}(dx_1 \wedge \cdots \wedge dt) \\ & - z\frac{\partial f}{\partial t} \pounds_{\frac{\partial}{\partial z}}(dx_1 \wedge \cdots \wedge dt) - d\left(z\frac{\partial f}{\partial t}\right) \wedge \iota_{\frac{\partial}{\partial z}}(dx_1 \wedge \cdots \wedge dt) \end{aligned} \tag{4.24}$$

In (4.24), we have

$$\begin{aligned} \pounds_{\frac{\partial}{\partial y_i}}(dx_1 \wedge \ldots \wedge dt) &= \sum_{j=1}^{n-1} dx_1 \wedge \cdots \wedge dy_{j-1} \wedge \left(\pounds_{\frac{\partial}{\partial y_i}} dx_j\right) \wedge dy_j \wedge \cdots \wedge dz \wedge dt \\ &+ \sum_{j=1}^{n-1} dx_1 \wedge \cdots \wedge dy_{j-1} \wedge dx_j \wedge \left(\pounds_{\frac{\partial}{\partial y_i}} dy_j\right) \wedge \cdots \wedge dz \wedge dt \\ &+ dx_1 \wedge \cdots \wedge dy_{n-1} \wedge \left(\pounds_{\frac{\partial}{\partial y_i}} dz\right) \wedge dt + dx_1 \wedge \cdots \wedge dy_{n-1} \wedge dz \wedge \left(\pounds_{\frac{\partial}{\partial y_i}} dt\right) \\ &= 0 \end{aligned}$$

since $\pounds \circ d = d \circ \pounds$. Similarly we find

$$0 = \pounds_{\frac{\partial}{\partial x_i}}(dx_1 \wedge \cdots \wedge dt) = \pounds_{\frac{\partial}{\partial t}}(dx_1 \wedge \cdots \wedge dt) = \pounds_{\frac{\partial}{\partial z}}(dx_1 \wedge \cdots \wedge dt).$$

Next,

$$\iota_{\frac{\partial}{\partial y_i}}(dx_1 \wedge \cdots \wedge dt) = -dx_1 \wedge \cdots \wedge dy_{i-1} \wedge dx_i \wedge dx_{i+1} \wedge dy_{i+1} \wedge \cdots \wedge dz \wedge dt,$$

hence

$$d\left(\frac{\partial f}{\partial x_i}\right)\wedge\iota_{\frac{\partial}{\partial y_i}}(dx_1\wedge\cdots\wedge dt)=\left(\frac{\partial^2 f}{\partial y_i\partial x_i}dy_i\right)\wedge(-dx_1\wedge\cdots\wedge dy_{i-1}\wedge dx_i\wedge dx_{i+1}\wedge\cdots\wedge dt)$$
$$=\frac{\partial^2 f}{\partial y_i\partial x_i}dx_1\wedge\cdots\wedge dt.$$

Similarly,

$$\iota_{\frac{\partial}{\partial x_i}}(dx_1\wedge\cdots\wedge dt)=dx_1\wedge\cdots\wedge dy_{i-1}\wedge dy_i\wedge dx_{i+1}\wedge\cdots\wedge dz\wedge dt,$$

hence

$$d\left(\frac{\partial f}{\partial y_i}\right)\wedge\iota_{\frac{\partial}{\partial x_i}}(dx_1\wedge\cdots\wedge dt)=\left(\frac{\partial^2 f}{\partial x_i\partial y_i}dx_i\right)\wedge dx_1\wedge\cdots\wedge dy_{i-1}\wedge dy_i\wedge dx_{i+1}\wedge\cdots\wedge dt$$
$$=\frac{\partial^2 f}{\partial x_i\partial y_i}dx_1\wedge\cdots\wedge dt.$$

Also,

$$\iota_{\frac{\partial}{\partial t}}(dx_1\wedge\cdots\wedge dt)=-dx_1\wedge\cdots\wedge dz,$$

hence

$$d\left(z\frac{\partial f}{\partial z}\right)\wedge\iota_{\frac{\partial}{\partial t}}(dx_1\wedge\cdots\wedge dt)=z\frac{\partial^2 f}{\partial t\partial z}dx_1\wedge\cdots\wedge dt.$$

Finally,

$$\iota_{\frac{\partial}{\partial z}}(dx_1\wedge\cdots\wedge dt)=dx_1\wedge\cdots\wedge dy_{n-1}\wedge dt$$

hence

$$d\left(z\frac{\partial f}{\partial t}\right)\wedge\iota_{\frac{\partial}{\partial z}}(dx_1\wedge\cdots\wedge dt)=\left(\frac{\partial f}{\partial t}+z\frac{\partial^2 f}{\partial z\partial t}\right)dx_1\wedge\cdots\wedge dt.$$

In conclusion,

$$\pounds_{X_f}\Omega=\sum_{i=1}^{n-1}\left[\frac{\partial^2 f}{\partial y_i\partial x_i}dx_1\wedge\cdots\wedge dt-\frac{\partial^2 f}{\partial x_i\partial y_i}dx_1\wedge\cdots\wedge dt\right]$$
$$+z\frac{\partial^2 f}{\partial t\partial z}dx_1\wedge\cdots\wedge dt-\left(\frac{\partial f}{\partial t}+z\frac{\partial^2 f}{\partial z\partial t}\right)dx_1\wedge\cdots\wedge dt$$
$$=-\frac{\partial f}{\partial t}dx_1\wedge\cdots\wedge dt$$

due to equality of the mixed partial derivatives. This shows that

$$X_\Pi^\Omega(f)=\frac{\pounds_{X_f}\Omega}{\Omega}=-\frac{\partial}{\partial t}(f).$$

Hence $X_\Pi^\Omega=-\partial/\partial t$. □

Proposition 4.2.15. *The modular vector field of a b-symplectic manifold (M,Z) is tangent to Z and transverse to the symplectic leaves inside Z, regardless of the volume form considered on M.*

Proof. Around a point $p \in Z$, we will work in the local coordinates mentioned in Lemma 4.2.14. With respect to the volume form Ω mentioned above, we have

$$X_\Pi^\Omega = -\partial/\partial t.$$

Note that Z is given near p by $z = 0$; so it has coordinates $(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, t)$ whence X_Π^Ω is tangent to Z. Moreover, the leaves of Z integrate $\mathrm{Ker}(dt)$ (as argued in Remark 4.2.11), hence they are the level sets of the t-coordinate. Therefore, X_Π^Ω is transverse to the leaves of Z, that is

$$T_qL + \mathrm{span}\left(X_\Pi^\Omega(q)\right) = T_qZ$$

for $q \in Z$ and L the leaf of Z through q. If we consider another volume form Ω', then we have $\Omega' = H\Omega$ for some non-vanishing function H defined near p. Proposition 2.9.6 then shows

$$X_\Pi^{\Omega'} = X_\Pi^\Omega - X_{\log|H|},$$

so that the modular vector field changes by a Hamiltonian vector field. Since $Z \subset M$ is a Poisson submanifold, Hamiltonian vector fields on M are tangent to Z at points $q \in Z$ (see Proposition 2.10.2). So $X_\Pi^{\Omega'}$ is still tangent to Z. Moreover, Hamiltonian vector fields are tangent to the symplectic leaves of M, and in particular to those inside Z. Therefore, $X_\Pi^{\Omega'}$ is still transverse to the leaves of Z. □

The existence of a Poisson vector field transverse to the leaves is rather special and useful. We will exploit this property later.

4.3 Cohomology theories for b-manifolds

We now discuss some cohomology theories for b-manifolds, and the relations between them. On a b-manifold (M, Z), we can talk about the usual cohomology theories for the underlying manifold M, such as de Rham cohomology and Poisson cohomology. However, we can also consider the complexes of b-forms and b-multivector fields, and study the corresponding cohomology theories. We will obtain cohomological obstructions for the existence of a b-symplectic structure, similar to those in symplectic geometry. Since log-symplectic structures can be considered dually as b-symplectic structures, these obstructions can be used to rule out the existence of a log-symplectic structure on certain manifolds.

4.3.1 De Rham cohomology and b-cohomology

Recall that on a b-manifold (M, Z) we have the complex of b-differential forms $\left({}^b\Omega^\bullet(M), {}^bd\right)$, where ${}^bd$ is the b-de Rham differential. The corresponding cohomology groups, denoted by ${}^bH^\bullet(M)$, are the b-de Rham cohomology groups, or b-cohomology groups for short. It turns out that the b-cohomology groups of a b-manifold (M, Z) are computable in terms of its ordinary de Rham cohomology groups.

Theorem 4.3.1 (b-Mazzeo-Melrose). *On a b-manifold (M, Z), we have the following decomposition for b-cohomology:*

$${}^bH^\bullet(M) \cong H^\bullet(M) \oplus H^{\bullet-1}(Z).$$

Proof. Fix an adapted distance function λ as in Lemma 4.1.16. We first show that there is a canonical short exact sequence of complexes

$$0 \longrightarrow \Omega^\bullet(M) \xrightarrow{i} {}^b\Omega^\bullet(M) \xrightarrow{\iota_\xi} \Omega^{\bullet-1}(Z) \longrightarrow 0, \tag{4.25}$$

where $i : \Omega^k(M) \to {}^b\Omega^k(M) : \omega \mapsto \omega$ is the inclusion map, and ι_ξ is contraction with the normal b-vector field

$$\iota_\xi : {}^b\Omega^k(M) \to \Omega^{k-1}(Z) : \omega = \alpha + d\log(\lambda) \wedge p^*(\theta) \mapsto \iota_\xi(\omega|_Z) = \theta.$$

Let us check that the maps involved are chain maps. For $\alpha + d\log(\lambda) \wedge p^*(\theta) \in {}^b\Omega^k(M)$ and $\omega \in \Omega^k(M)$, we have

- ${}^bd(i(\omega)) = {}^bd(\omega + d\log(\lambda) \wedge 0) = d\omega = i(d\omega)$.
- $d(\iota_\xi(\alpha + d\log(\lambda) \wedge p^*(\theta))) = d\theta$, whereas

  $$\iota_\xi\big({}^bd(\alpha+d\log(\lambda)\wedge p^*(\theta))\big) = \iota_\xi\big(d\alpha+d\log(\lambda)\wedge d(p^*(\theta))\big) = \iota_\xi\big(d\alpha+d\log(\lambda)\wedge p^*(d\theta)\big) = d\theta.$$

Next, we show that for each $k \in \mathbb{N}$, the sequence

$$0 \longrightarrow \Omega^k(M) \xrightarrow{\ i\ } {}^b\Omega^k(M) \xrightarrow{\ \iota_\xi\ } \Omega^{k-1}(Z) \longrightarrow 0$$

is exact.

- The inclusion map i is injective: if $i(\omega) = i(\omega')$ for some $\omega, \omega' \in \Omega^k(M)$, then $\omega = \omega'$ as b-forms. Hence $\omega = \omega'$ on $M \setminus Z$ as de Rham forms. Since $M \setminus Z \subset M$ is dense and ω, ω' are continuous, we get equality $\omega = \omega'$ on all of M.
- Clearly ι_ξ is surjective, for if $\theta \in \Omega^{k-1}(Z)$ is given then $\iota_\xi\big(d\log(\lambda) \wedge p^*(\theta)\big) = \theta$.
- For $\omega \in \Omega^k(M)$, we have $\iota_\xi(i(\omega)) = \iota_\xi(\omega) = 0$. Indeed, by convention we have $\omega_p \in T_p^*Z$ for all $p \in Z$, and we have seen that $T_p^*Z = \langle \xi_p \rangle^0$. Hence $\mathrm{Im}(i) \subset \mathrm{Ker}(\iota_\xi)$. Conversely, if $\alpha + d\log(\lambda) \wedge p^*(\theta) \in \mathrm{Ker}(\iota_\xi)$ then $\theta = 0$, so that $\alpha + d\log(\lambda) \wedge p^*(\theta) = \alpha = i(\alpha)$.

Hence the sequence (4.25) is exact, and since ξ is canonical, so is the sequence. Next, we show that the sequence (4.25) splits. A splitting is given by

$$\sigma : \Omega^{\bullet-1}(Z) \to {}^b\Omega^\bullet(M) : \theta \mapsto d\log(\lambda) \wedge p^*(\theta).$$

Then σ is a chain map because

$$\sigma(d\theta) = d\log(\lambda) \wedge p^*(d\theta) = d\log(\lambda) \wedge d\big(p^*(\theta)\big) = {}^bd\big(d\log(\lambda) \wedge p^*(\theta)\big) = {}^bd(\sigma(\theta))$$

and clearly it splits the sequence as

$$\left(\iota_\xi \circ \sigma\right)(\theta) = \iota_\xi\left(d\log(\lambda) \wedge p^*(\theta)\right) = \theta.$$

It is well-known that a short exact sequence of cochain complexes induces a long exact sequence in cohomology

$$\cdots \xrightarrow{\ \delta\ } H^k(M) \xrightarrow{\ \bar{i}\ } {}^bH^k(M) \xrightarrow{\ \overline{\iota_\xi}\ } H^{k-1}(Z) \xrightarrow{\ \delta\ } H^{k+1}(M) \longrightarrow \cdots \tag{4.26}$$

where δ is the connecting homomorphism. The long exact sequence (4.26) gives rise to short exact sequences of the form

$$0 \longrightarrow \mathrm{Coker}(\delta) \xrightarrow{\ \bar{i}\ } {}^bH^k(M) \xrightarrow{\ \overline{\iota_\xi}\ } \mathrm{Im}(\overline{\iota_\xi}) \longrightarrow 0. \tag{4.27}$$

Since $\iota_\xi \circ \sigma = \mathrm{Id}_{\Omega^{k-1}(Z)}$ and passing to cohomology is a covariant functor, we also have

$$\overline{\iota_\xi} \circ \overline{\sigma} = \mathrm{Id}_{H^{k-1}(Z)}.$$

This implies that $\overline{\iota_\xi}$ is surjective and that $\overline{\sigma}$ is injective. By exactness, the connecting homomorphisms δ in (4.26) are then zero maps. Hence the short exact sequence (4.27) becomes

$$0 \longrightarrow H^k(M) \xrightarrow{\bar{i}} {}^bH^k(M) \xrightarrow{\overline{\iota_\xi}} H^{k-1}(Z) \longrightarrow 0.$$

Since $\overline{\iota_\xi} \circ \overline{\sigma} = \mathrm{Id}_{H^{k-1}(Z)}$, this sequence splits (as does any short exact sequence of vector spaces). We conclude that

$${}^bH^k(M) = \bar{i}(H^k(M)) \oplus \overline{\sigma}(H^{k-1}(Z)) \cong H^k(M) \oplus H^{k-1}(Z),$$

where the last isomorphism holds by injectivity of $\bar{i}$ and $\overline{\sigma}$. □

Example 4.3.2. For $(M, Z) = (S^2, S^1)$, we have

- ${}^bH^0(S^2) = H^0(S^2) = \mathbb{R}$.
- ${}^bH^1(S^2) = H^1(S^2) \oplus H^0(S^1) = \mathbb{R}$.
- ${}^bH^2(S^2) = H^2(S^2) \oplus H^1(S^1) = \mathbb{R} \oplus \mathbb{R}$.
- ${}^bH^k(S^2) = 0$ for all $k \geq 3$.

We obtain some obstructions to the existence of a b-symplectic structure. Proposition 1.2.8 shows that the second de Rham cohomology group of a compact symplectic manifold is nonzero. The b-analog of this statement is also true.

Proposition 4.3.3. *For a compact b-symplectic manifold (M, Z), we have $H^1(Z) \neq 0$ and consequently ${}^bH^2(M) \neq 0$.*

Proof. Let $\omega = \alpha + d\log(\lambda) \wedge p^*(\theta)$ be a b-symplectic form on (M, Z), where $\alpha \in \Omega^2(M)$ and $\theta \in \Omega^1(Z)$. By Lemma 4.2.5, we know that θ is closed and nowhere vanishing. Let Π be the log-symplectic structure dual to ω. Then $Z \subset M$ is closed, being the vanishing locus of Π^n. Since M is compact, this implies that Z is compact as well. If we would have $H^1(Z) = 0$, then θ would be exact: $\theta = dg$ for some function $g \in C^\infty(Z)$. Being a continuous function on a compact domain, the function g has maximum and minimum points on Z, at which $\theta = dg$ necessarily vanishes. This is impossible. Then, by Theorem 4.3.1 we have

$${}^bH^2(M) \cong H^2(M) \oplus H^1(Z) \neq 0.$$

□

Above proposition shows for instance that (S^4, S^3) cannot be log-symplectic.

Proposition 4.3.4. *For a compact b-symplectic manifold (M^{2n}, Z) with $n \geq 2$, we have $H^2(Z) \neq 0$ and consequently ${}^bH^3(M) \neq 0$.*

Proof. Let $\omega = \alpha + d\log(\lambda) \wedge p^*(\theta)$ be a b-symplectic form on (M^{2n}, Z), where $\alpha \in \Omega^2(M)$ and $\theta \in \Omega^1(Z)$. Denote by $i : Z \hookrightarrow M$ the inclusion map. By Lemma 4.2.5, we know that $i^*\alpha$ and θ are closed and that $(i^*\alpha)^{n-1} \wedge \theta$ is a volume form on Z. Assume by contradiction that $H^2(Z) = 0$. Then $i^*\alpha = d\mu$ for some one-form $\mu \in \Omega^1(Z)$. Using compactness of M and Stokes' theorem, we would then get

$$\begin{aligned} 0 \neq \mathrm{Vol}(Z) &= \int_Z (i^*\alpha)^{n-1} \wedge \theta = \int_Z (d\mu)^{n-1} \wedge \theta = \int_Z d\big(\mu \wedge (d\mu)^{n-2} \wedge \theta\big) \\ &= \int_{\partial Z} \mu \wedge (d\mu)^{n-2} \wedge \theta = 0, \end{aligned}$$

where the last equality holds since $\partial Z = \emptyset$. Thus $H^2(Z)$ must be nonzero, and Theorem 4.3.1 gives ${}^bH^3(M) \neq 0$. □

There are also obstructions to the existence of a log-symplectic structure in the usual de Rham cohomology. Their proofs are technical, so we will content ourselves by just stating the results.

Theorem 4.3.5 ([MO2]). *Let (M^{2n}, Π) be a compact log-symplectic manifold. Then there exists a class $c \in H^2(M)$ such that $c^{n-1} \in H^{2n-2}(M)$ is nonzero.*

Compare this to Proposition 1.2.8 in symplectic geometry, which states that for a symplectic manifold M^{2n}, there exists a class $c \in H^2(M)$ such that $c^n \in H^{2n}(M)$ is nonzero. For log-symplectic structures, that property does not hold in general: there are log-symplectic manifolds that are compact, connected and non-orientable ($\mathbb{R}P^2$ is such an example) and their top de Rham cohomology group vanishes altogether. But Theorem 4.3.5 shows that log-symplectic structures are only a little shy of satisfying this property.

One can use Theorem 4.3.5 to determine which spheres S^{2n} for $n > 0$ are log-symplectic. We know that S^2 is log-symplectic, by Example 3.1.5. Higher-dimensional spheres cannot be log-symplectic, by Theorem 4.3.5. So a sphere S^{2n} is log-symplectic if and only if it is symplectic.

The following obstruction is more contrastive with symplectic geometry.

Theorem 4.3.6 ([Cav]). *If a compact oriented manifold M^{2n}, with $n > 1$, admits a bona fide log-symplectic structure, then there are classes $a, b \in H^2(M)$ such that $a^{n-1}b \neq 0$ and $b^2 \neq 0$.*

This theorem shows, for instance, that $\mathbb{C}P^n$ does not admit a *bona fide* log-symplectic structure when $n > 1$. Note however that $\mathbb{C}P^n$ is symplectic; this can be obtained by symplectic reduction, for instance.

4.3.2 Poisson cohomology and b-Poisson cohomology

Suppose we are given a b-manifold (M, Z) and a Poisson structure Π on M such that $Z \subset M$ is a Poisson submanifold. The Poisson bivector Π induces a differential $d_\Pi = [\Pi, \cdot]$ on the graded algebra of multivector fields $\mathfrak{X}^\bullet(M)$. The cohomology of the complex

$$\cdots \longrightarrow \mathfrak{X}^{k-1}(M) \xrightarrow{d_\Pi} \mathfrak{X}^k(M) \xrightarrow{d_\Pi} \mathfrak{X}^{k+1}(M) \longrightarrow \cdots$$

is the Poisson cohomology $H^\bullet_\Pi(M)$ of M. We can also consider the space of b-multivector fields ${}^b\mathfrak{X}^\bullet(M) = \Gamma\left(\wedge^\bullet({}^bTM)\right)$, which consists of the multivector fields on M that are tangent to Z. Note that $\left({}^b\mathfrak{X}^\bullet(M), d_\Pi\right)$ is a subcomplex of $(\mathfrak{X}^\bullet(M), d_\Pi)$. Indeed, given $\xi \in {}^b\mathfrak{X}^k(M)$, we have show that $[\Pi, \xi]$ is tangent to Z. Since $Z \subset M$ is a Poisson submanifold, we have that Π is tangent to Z. Also ξ is tangent to Z, hence denoting by $i : Z \hookrightarrow M$ the inclusion map, we have that $\xi|_Z$ and ξ are i-related and that $\Pi|_Z$ and Π are i-related. By Lemma 8.3.2 in the appendix, also $[\Pi|_Z, \xi|_Z]$ and $[\Pi, \xi]$ are i-related. In particular, $[\Pi, \xi]$ is tangent to Z. Hence $\left({}^b\mathfrak{X}^\bullet(M), d_\Pi\right)$ is a subcomplex of $(\mathfrak{X}^\bullet(M), d_\Pi)$, or stated otherwise, the inclusion ${}^b\mathfrak{X}^\bullet(M) \subset \mathfrak{X}^\bullet(M)$ is a chain map. The cohomology of the complex

$$\cdots \longrightarrow {}^b\mathfrak{X}^{k-1}(M) \xrightarrow{d_\Pi} {}^b\mathfrak{X}^k(M) \xrightarrow{d_\Pi} {}^b\mathfrak{X}^{k+1}(M) \longrightarrow \cdots$$

is the b-Poisson cohomology ${}^bH^\bullet_\Pi(M)$ of M. Similar to what happens in the symplectic case, we have

Theorem 4.3.7. *Let (M^{2n}, Z) be a b-symplectic manifold, with corresponding log-symplectic structure Π. Then the b-Poisson cohomology ${}^bH^\bullet_\Pi(M)$ is isomorphic to the b-de Rham cohomology ${}^bH^\bullet(M)$.*

Proof. Considering Π as a section of $\wedge^2({}^bTM)$, we have a morphism $\Pi^\sharp : {}^bT^*M \to {}^bTM$. Taking exterior powers, we extend it to a map $\wedge^k({}^bT^*M) \to \wedge^k({}^bTM)$. On the level of sections, this is a $C^\infty(M)$-linear map, given by

$$ {}^b\Omega^k(M) \to {}^b\mathfrak{X}^k(M) : \alpha_1 \wedge \cdots \wedge \alpha_k \mapsto \Pi^\sharp(\alpha_1) \wedge \cdots \wedge \Pi^\sharp(\alpha_k). $$

We will denote this map by $\Pi^\sharp$ as well. By convention, $\Pi^\sharp(f) = f$ for all $f \in C^\infty(M) = {}^b\Omega^0(M)$.
Claim: Up to sign, the map $\Pi^\sharp : {}^b\Omega^k(M) \to {}^b\mathfrak{X}^k(M)$ is a chain map. That is,

$$ \Pi^\sharp({}^bd\eta) = -d_\Pi(\Pi^\sharp(\eta)) \qquad \text{for all } \eta \in {}^b\Omega^k(M). \tag{4.28} $$

We prove the claim by induction on the degree k of η. If $\eta \in C^\infty(M)$, then

$$ -d_\Pi(\Pi^\sharp(\eta)) = -d_\Pi(\eta) = -[\Pi, \eta] = \iota_{d\eta}\Pi = \Pi^\sharp(d\eta) = \Pi^\sharp({}^bd\eta), $$

where we used Lemma 8.3.1 in the appendix. Now let $\eta \in {}^b\Omega^1(M)$. It is enough to check (4.28) in coordinates near Z (away from Z, the equality (4.28) is true by Lemma 2.8.4). We can choose coordinates such that

$$ \Pi = \frac{\partial}{\partial x_1} \wedge \left(y_1 \frac{\partial}{\partial y_1} \right) + \sum_{i=2}^n \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}, $$

and in these coordinates,

$$ \eta = f \frac{dy_1}{y_1} + g dx_1 + \sum_{i=2}^n (f_i dy_i + g_i dx_i). $$

On one hand, we get

$$ \begin{aligned} \Pi^\sharp({}^bd\eta) &= \Pi^\sharp \left(df \wedge \frac{dy_1}{y_1} + dg \wedge dx_1 + \sum_{i=2}^n (df_i \wedge dy_i + dg_i \wedge dx_i) \right) \\ &= \Pi^\sharp(df) \wedge \Pi^\sharp \left(\frac{dy_1}{y_1} \right) + \Pi^\sharp(dg) \wedge \Pi^\sharp(dx_1) + \sum_{i=2}^n (\Pi^\sharp(df_i) \wedge \Pi^\sharp(dy_i) + \Pi^\sharp(dg_i) \wedge \Pi^\sharp(dx_i)) \\ &= -\Pi^\sharp(df) \wedge \left(\frac{\partial}{\partial x_1} \right) + \Pi^\sharp(dg) \wedge \left(y_1 \frac{\partial}{\partial y_1} \right) - \sum_{i=2}^n \Pi^\sharp(df_i) \wedge \frac{\partial}{\partial x_i} + \sum_{i=2}^n \Pi^\sharp(dg_i) \wedge \frac{\partial}{\partial y_i}. \end{aligned} $$

On the other hand,

$$ \begin{aligned} -d_\Pi(\Pi^\sharp(\eta)) &= -d_\Pi \left(f \Pi^\sharp \left(\frac{dy_1}{y_1} \right) \right) - d_\Pi(g\Pi^\sharp(dx_1)) - \sum_{i=2}^n d_\Pi(f_i \Pi^\sharp(dy_i) + g_i \Pi^\sharp(dx_i)) \\ &= d_\Pi \left(f \frac{\partial}{\partial x_1} \right) - d_\Pi \left(g y_1 \frac{\partial}{\partial y_1} \right) + \sum_{i=2}^n d_\Pi \left(f_i \frac{\partial}{\partial x_i} \right) - \sum_{i=2}^n d_\Pi \left(g_i \frac{\partial}{\partial y_i} \right). \end{aligned} $$

Here

$$ \begin{aligned} d_\Pi \left(f \frac{\partial}{\partial x_1} \right) = \left[\Pi, f \frac{\partial}{\partial x_1} \right] &= [\Pi, f] \wedge \frac{\partial}{\partial x_1} + f \left[\Pi, \frac{\partial}{\partial x_1} \right] \\ &= -\Pi^\sharp(df) \wedge \frac{\partial}{\partial x_1} - f \pounds_{\frac{\partial}{\partial x_1}} \Pi \\ &= -\Pi^\sharp(df) \wedge \frac{\partial}{\partial x_1}, \end{aligned} $$

$$
\begin{aligned}
-d_\Pi\left(gy_1\frac{\partial}{\partial y_1}\right) &= -\left[\Pi, gy_1\frac{\partial}{\partial y_1}\right] = -\left([\Pi,g]\wedge y_1\frac{\partial}{\partial y_1} + g\left[\Pi, y_1\frac{\partial}{\partial y_1}\right]\right)\\
&= \Pi^\sharp(dg)\wedge y_1\frac{\partial}{\partial y_1} + g\pounds_{y_1\frac{\partial}{\partial y_1}}\Pi\\
&= \Pi^\sharp(dg)\wedge y_1\frac{\partial}{\partial y_1},
\end{aligned}
$$

$$
\begin{aligned}
-d_\Pi\left(g_i\frac{\partial}{\partial y_i}\right) &= -\left[\Pi, g_i\frac{\partial}{\partial y_i}\right] = -[\Pi,g_i]\wedge\frac{\partial}{\partial y_i} - g_i\left[\Pi,\frac{\partial}{\partial y_i}\right]\\
&= \Pi^\sharp(dg_i)\wedge\frac{\partial}{\partial y_i} + g_i\pounds_{\frac{\partial}{\partial y_i}}\Pi\\
&= \Pi^\sharp(dg_i)\wedge\frac{\partial}{\partial y_i},
\end{aligned}
$$

and

$$
d_\Pi\left(f_i\frac{\partial}{\partial x_i}\right) = -\Pi^\sharp(df_i)\wedge\frac{\partial}{\partial x_i}.
$$

So (4.28) is true for b-one forms. Finally, if the formula holds for $\eta\in{}^b\Omega^p(M)$ and $\mu\in{}^b\Omega^q(M)$, then it also holds for $\eta\wedge\mu$. Indeed, using that ${}^bd$ is a degree 1 derivation of $\wedge$, we have

$$
\begin{aligned}
\Pi^\sharp\big({}^bd(\eta\wedge\mu)\big) &= \Pi^\sharp\big({}^bd\eta\wedge\mu + (-1)^p\eta\wedge{}^bd\mu\big) = \Pi^\sharp\big({}^bd\eta\big)\wedge\Pi^\sharp(\mu) + (-1)^p\Pi^\sharp(\eta)\wedge\Pi^\sharp\big({}^bd\mu\big)\\
&= -d_\Pi\big(\Pi^\sharp(\eta)\big)\wedge\Pi^\sharp(\mu) - (-1)^p\Pi^\sharp(\eta)\wedge d_\Pi\big(\Pi^\sharp(\mu)\big)\\
&= -[\Pi,\Pi^\sharp(\eta)]\wedge\Pi^\sharp(\mu) - (-1)^p\Pi^\sharp(\eta)\wedge[\Pi,\Pi^\sharp(\mu)]\\
&= -[\Pi,\Pi^\sharp(\eta)\wedge\Pi^\sharp(\mu)]\\
&= -d_\Pi\big(\Pi^\sharp(\eta\wedge\mu)\big).
\end{aligned}
$$

It follows that we have induced morphisms between cohomology groups

$$
[\Pi^\sharp]: {}^bH^k(M)\to{}^bH^k_\Pi(M): [\eta]\mapsto[\Pi^\sharp(\eta)].
$$

Since $\Pi\in\Gamma\left(\wedge^2({}^bTM)\right)$ is non-degenerate, we have that $\Pi^\sharp: {}^bT^*M\to{}^bTM$ is a bundle isomorphism. Hence the same holds for its exterior powers $\Pi^\sharp:\wedge^k\left({}^bT^*M\right)\to\wedge^k\left({}^bTM\right)$. On the chain level, we get isomorphisms of $C^\infty(M)$-modules $\Pi^\sharp: {}^b\Omega^k(M)\to{}^b\mathfrak{X}^k(M)$. Since passing to cohomology is functorial, it follows that the induced maps on cohomology

$$
[\Pi^\sharp]: {}^bH^k(M)\to{}^bH^k_\Pi(M)
$$

are isomorphisms. □

There are more conceptual ways to see that the claim in above proof is true. For instance, we know by Lemma 2.8.4 that $\Pi^\sharp$ is a chain map (up to sign) on $M\setminus Z$, so that the claim follows from continuity arguments.

A more advanced approach is the following. It is well-known that for any Poisson manifold (N,Π), the cotangent bundle T^*N is a Lie algebroid with anchor map $\Pi^\sharp: T^*N\to TN$ and Lie bracket $[df,dg]=d\{f,g\}$. Trivially, also TN is a Lie algebroid. By a general fact in the theory of Lie algebroids, the anchor map $\Pi^\sharp: T^*N\to TN$ is a Lie algebroid morphism. Therefore, wedges of its dual give a chain map, where $\Gamma(\wedge^\bullet TN)$ and $\Gamma(\wedge^\bullet T^*N)$ are endowed with the induced Lie algebroid differentials, which are the usual Lichnerowicz differential d_Π and the usual de Rham differential d, respectively. Since $\Pi^\sharp$ is skew-symmetric, its dual map is $-\Pi^\sharp$, and therefore we obtain a chain map

$$\wedge^\bullet(-\Pi^\sharp) : \big(\Gamma(\wedge^\bullet T^*N), d\big) \to \big(\Gamma(\wedge^\bullet TN), d_\Pi\big).$$

Applying this to $N = M \setminus Z$, the claim again follows from a continuity argument.

For a log-symplectic manifold (M, Z), the b-Poisson cohomology does not give any additional information: the b-Poisson cohomology groups are isomorphic to the Poisson cohomology groups of M.

Theorem 4.3.8. *Let (M, Z, Π) be a log-symplectic manifold. The inclusion ${}^b\mathfrak{X}^\bullet(M) \subset \mathfrak{X}^\bullet(M)$ induces an isomorphism in cohomology, i.e. the Poisson cohomology is isomorphic to the b-Poisson cohomology:*

$$H^\bullet_\Pi(M) \cong {}^bH^\bullet_\Pi(M).$$

The proof of Theorem 4.3.8 relies on the following Poisson version of Cartan's magic formula: If Π is a Poisson bivector and β is a closed one-form, then we have the equality

$$\iota_\beta \circ d_\Pi + d_\Pi \circ \iota_\beta = \mathcal{L}_{\Pi^\sharp(\beta)} \tag{4.29}$$

on multivector fields. The proof of (4.29) is a rather painful calculation that can be found in the appendix.

Proof. (*of Theorem 4.3.8*) We will construct linear maps $h : \mathfrak{X}^\bullet(M) \to \mathfrak{X}^{\bullet-1}(M)$ such that we obtain a linear map

$$\zeta : \mathfrak{X}^\bullet(M) \to {}^b\mathfrak{X}^\bullet(M) : w \mapsto w + (d_\Pi \circ h)(w) + (h \circ d_\Pi)(w). \tag{4.30}$$

This map ζ is a chain map between the complexes $(\mathfrak{X}^\bullet(M), d_\Pi)$ and $\big({}^b\mathfrak{X}^\bullet(M), d_\Pi\big)$ since

$$(d_\Pi \circ \zeta)(w) = d_\Pi(w) + (d_\Pi \circ h \circ d_\Pi)(w),$$
$$(\zeta \circ d_\Pi)(w) = d_\Pi(w) + (d_\Pi \circ h \circ d_\Pi)(w).$$

Hence it induces a map in cohomology

$$[\zeta] : H^\bullet_\Pi(M) \to {}^bH^\bullet_\Pi(M) : [w] \mapsto [\zeta(w)] = [w].$$

On the other hand, we have that the inclusion $i : {}^b\mathfrak{X}^\bullet(M) \hookrightarrow \mathfrak{X}^\bullet(M)$ is a chain map since $\big({}^b\mathfrak{X}^\bullet(M), d_\Pi\big)$ is a subcomplex of $(\mathfrak{X}^\bullet(M), d_\Pi)$, whence we get a map

$$[i] : {}^bH^\bullet_\Pi(M) \to H^\bullet_\Pi(M) : [v] \mapsto [v].$$

Clearly, the maps $[\zeta]$ and $[i]$ are inverses of each other, which then implies the conclusion that

$$[i] : {}^bH^\bullet_\Pi(M) \xrightarrow{\sim} H^\bullet_\Pi(M).$$

Let E be a tubular neighborhood of Z in M, with projection $p : E \to Z$. Let $E' \subset E$ be a smaller tubular neighborhood of Z, and let χ be a smooth function supported in E, such that $\chi|_{E'} = 1$. If ω is the b-symplectic form corresponding with Π, then we know that with ω comes a canonical closed 1-form $\theta \in \Omega^1(Z)$ which defines the symplectic foliation of Z (see Lemma 4.2.5 and Remark 4.2.11). We now define the operators h by

$$h : \mathfrak{X}^\bullet(M) \to \mathfrak{X}^{\bullet-1}(M) : h(w) := \iota_{-p^*(\theta)}(\chi w).$$

These are clearly well-defined and linear. It remains to show that $\zeta(w)$ is tangent to Z for all $w \in \mathfrak{X}^\bullet(M)$, where ζ is defined in (4.30). To do this, we can work in E' where $\chi \equiv 1$ and hence we must show that

$$w + \left(d_\Pi \circ \iota_{-p^*(\theta)}\right)(w) + \left(\iota_{-p^*(\theta)} \circ d_\Pi\right)(w) \in {}^b\mathfrak{X}^k(M) \tag{4.31}$$

for $w \in \mathfrak{X}^k(M)$. Note that $p^*(\theta)$ is closed, so that the Cartan formula (4.29) holds:

$$\iota_{-p^*(\theta)} \circ d_\Pi + d_\Pi \circ \iota_{-p^*(\theta)} = \pounds_{-\Pi^\sharp(p^*(\theta))}.$$

Hence to show (4.31), it suffices to check that $w + \pounds_{-\Pi^\sharp(p^*(\theta))} w \in {}^b\mathfrak{X}^k(M)$. First we note that, since $Z \subset M$ is a Poisson submanifold, $\mathrm{Im}(\Pi_p^\sharp) \subset T_pZ$ for all $p \in Z$, so that $\nu := -\Pi^\sharp(p^*(\theta))$ is a b-vector field. Next, if ξ denotes the normal b-vector field of (M, Z) then we have seen that

$$\omega^\flat\Big|_Z(\xi) = \iota_\xi(\omega|_Z) = \theta.$$

On the other hand,

$$\omega^\flat\Big|_Z(\nu|_Z) = \omega^\flat(\nu)|_Z = \omega^\flat\big(-\Pi^\sharp(p^*(\theta))\big) = p^*(\theta)|_Z = i^*(p^*(\theta)) = (p \circ i)^*\theta = \theta,$$

where we used that $\omega^\flat = -(\Pi^\sharp)^{-1}$ (considering Π as a section of $\wedge^2({}^bTM)$), as well as the convention (4.7). Since $\omega^\flat$ is invertible (being a b-symplectic form), this then implies that $\nu|_Z = \xi$. With this information, we can now show that for all $w \in \mathfrak{X}^k(M)$:

$$w + \pounds_{-\Pi^\sharp(p^*(\theta))} w = w + \left[-\Pi^\sharp(p^*(\theta)), w\right] = w + [\nu, w] \in {}^b\mathfrak{X}^k(M).$$

Choose adapted coordinates $(x_1, \dots, x_n)$ such that Z is locally given by $x_1 = 0$. We can write

$$w = \sum_{i_1 < \cdots < i_k} w_{i_1 \cdots i_k} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}},$$
$$\nu = g_1 x_1 \frac{\partial}{\partial x_1} + \sum_{i=2}^n g_i \frac{\partial}{\partial x_i}.$$

Since $\nu|_Z = \xi = x_1 \frac{\partial}{\partial x_1}$, we must have that $g_1|_Z = 1$ and $x_1 | g_i$ for all $2 \le i \le n$. Hence we can write

$$\nu = \sum_{i=1}^n f_i x_1 \frac{\partial}{\partial x_i},$$

where $f_1|_Z = 1$. We get

$$\begin{aligned} w + [\nu, w] &= \sum_{i_1 < \cdots < i_k} w_{i_1 \cdots i_k} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}} + \pounds_{\sum_{j=1}^n f_j x_1 \frac{\partial}{\partial x_j}} \left(\sum_{i_1 < \cdots < i_k} w_{i_1 \cdots i_k} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}} \right) \\ &= \sum_{i_1 < \cdots < i_k} w_{i_1 \cdots i_k} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}} + \sum_{i_1 < \cdots < i_k} \left(\sum_{j=1}^n f_j x_1 \frac{\partial w_{i_1 \cdots i_k}}{\partial x_j} \right) \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}} \\ &\quad - \sum_{l=1}^k \sum_{i_1 < \cdots < i_k} w_{i_1 \cdots i_k} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{l-1}}} \wedge \pounds_{\frac{\partial}{\partial x_{i_l}}} \left(\sum_{j=1}^n f_j x_1 \frac{\partial}{\partial x_j} \right) \wedge \frac{\partial}{\partial x_{i_{l+1}}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_k}}. \end{aligned} \tag{4.32}$$

Here

- $\sum_{i_1<\cdots<i_k}\left(\sum_{j=1}^n f_j x_1 \frac{\partial w_{i_1\cdots i_k}}{\partial x_j}\right)\frac{\partial}{\partial x_{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_k}}$ is clearly a b-vector field.
- The last sum in (4.32) for $l=1$ gives

$$\begin{aligned}
&\sum_{i_1<\cdots<i_k} w_{i_1\cdots i_k}\pounds_{\frac{\partial}{\partial x_{i_1}}}\left(\sum_{j=1}^n f_j x_1 \frac{\partial}{\partial x_j}\right)\wedge\frac{\partial}{\partial x_{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_k}}\\
&=\sum_{i_1<\cdots<i_k} w_{i_1\cdots i_k}\left(\sum_{j=1}^n \frac{\partial(f_j x_1)}{\partial x_{i_1}}\frac{\partial}{\partial x_j}\right)\wedge\frac{\partial}{\partial x_{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_k}}\\
&=\sum_{i_1<\cdots<i_k} w_{i_1\cdots i_k}\left(\sum_{j=1}^n \left(\frac{\partial f_j}{\partial x_{i_1}}x_1+f_j\frac{\partial x_1}{\partial x_{i_1}}\right)\frac{\partial}{\partial x_j}\right)\wedge\frac{\partial}{\partial x_{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_k}},
\end{aligned}$$

  where

$$\sum_{i_1<\cdots<i_k} w_{i_1\cdots i_k}\left(\sum_{j=1}^n \frac{\partial f_j}{\partial x_{i_1}}x_1\frac{\partial}{\partial x_j}\right)\wedge\frac{\partial}{\partial x_{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_k}}$$

  is a b-vector field. So we are left with

$$\begin{aligned}
&\sum_{i_1<\cdots<i_k} w_{i_1\cdots i_k}\left(\sum_{j=1}^n f_j\frac{\partial x_1}{\partial x_{i_1}}\frac{\partial}{\partial x_j}\right)\wedge\frac{\partial}{\partial x_{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_k}}\\
&=\sum_{1<i_2<\cdots<i_k} w_{1i_2\cdots i_k}\left(\sum_{j=1}^n f_j\frac{\partial}{\partial x_j}\right)\wedge\frac{\partial}{\partial x_{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_k}}
\end{aligned}$$

- Note that for $l>1$:

$$\pounds_{\frac{\partial}{\partial x_{i_l}}}\left(\sum_{j=1}^n f_j x_1\frac{\partial}{\partial x_j}\right)=\sum_{j=1}^n\left(\frac{\partial f_j}{\partial x_{i_l}}x_1+f_j\frac{\partial x_1}{\partial x_{i_l}}\right)\frac{\partial}{\partial x_j}=\sum_{j=1}^n\frac{\partial f_j}{\partial x_{i_l}}x_1\frac{\partial}{\partial x_j},$$

  where the last equality holds since $i_l>1$ for $l>1$. Hence the last sum in (4.32) for $l>1$ gives a b-vector field.

So omitting all terms in (4.32) that are for sure b-vector fields, we are left with

$$\sum_{i_1<\cdots<i_k} w_{i_1\cdots i_k}\frac{\partial}{\partial x_{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_k}}-\sum_{1<i_2<\cdots<i_k} w_{1i_2\cdots i_k}\left(\sum_{j=1}^n f_j\frac{\partial}{\partial x_j}\right)\wedge\frac{\partial}{\partial x_{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_k}}. \quad (4.33)$$

To show that this is a b-vector field, we consider its terms containing $\partial/\partial x_1$:

$$\begin{aligned}
&\sum_{1<i_2<\cdots<i_k} w_{1i_2\cdots i_k}\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_k}}-\sum_{1<i_2<\cdots<i_k} w_{1i_2\cdots i_k}f_1\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_k}}\\
&=\sum_{1<i_2<\cdots<i_k}(w_{1i_2\cdots i_k}-w_{1i_2\cdots i_k}f_1)\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial x_{i_2}}\wedge\cdots\wedge\frac{\partial}{\partial x_{i_k}} \quad (4.34)
\end{aligned}$$

Since $f_1|_Z=1$, we have that (4.34) vanishes on Z, so that the coefficients $w_{1i_2\cdots i_k}-w_{1i_2\cdots i_k}f_1$ are smooth multiples of x_1. This shows that also the remaining terms (4.33) are b-vector fields. □

Recall that the inclusion ${}^b\mathfrak{X}^\bullet(M) \hookrightarrow \mathfrak{X}^\bullet(M)$ is induced by the anchor map $\rho : {}^bTM \to TM$, whereas the inclusion $\Omega^\bullet(M) \hookrightarrow {}^b\Omega^\bullet(M)$ is induced by its dual $\rho^* : T^*M \to {}^bT^*M$. Hence, in conclusion of this section, for a log-symplectic structure (M, Z, Π) we have a diagram of vector bundles and vector bundle maps

$$\begin{CD} {}^bTM @>{\rho}>> TM \\ @A{(\rho^{-1}(\Pi))^\sharp}AA @AA{\Pi^\sharp}A \\ {}^bT^*M @<<{\rho^*}< T^*M \end{CD} .$$

On the level of sections, it becomes a diagram of complexes with chain maps (up to sign)

$$\begin{CD} {}^b\mathfrak{X}^\bullet(M) @>>> \mathfrak{X}^\bullet(M) \\ @A{(\rho^{-1}(\Pi))^\sharp}AA @AA{\Pi^\sharp}A \\ {}^b\Omega^\bullet(M) @<<< \Omega^\bullet(M) \end{CD} ,$$

two of which induce an isomorphism in cohomology

$$\begin{CD} {}^bH^\bullet_\Pi(M) @>{\sim}>> H^\bullet_\Pi(M) \\ @A{\wr}AA @AAA \\ {}^bH^\bullet(M) @<<< H^\bullet(M) \end{CD} .$$

Chapter 5

The structure of log-symplectic manifolds near their singular loci

In light of the objectives of the thesis, this chapter contains the main results. It aims to describe log-symplectic structures (M, Z, Π) semilocally, in a neighborhood of the singular locus Z. Following [BOT, Section 4.1], we present a normal form model for log-symplectic structures (M, Z, Π), valid in a tubular neighborhood of Z. Next, we will address log-symplectic extensions of corank-one Poisson structures. In [GMP2], one obtained necessary and sufficient conditions for a corank-one Poisson structure Π_Z on Z to be induced by a log-symplectic structure. We will see to what extent such log-symplectic extensions are unique, presenting statements from [GMP2] complemented by some original observations and proofs.

5.1 Cosymplectic structures revisited

We first elaborate on the brief introduction to cosymplectic structures given in the previous chapter, as these will play a key role in what follows. Cosymplectic structures show up naturally when dealing with corank-one Poisson structures. This is demonstrated by the next theorem, which is mentioned in [MO], and which serves as a refinement of [GMP1, Proposition 18].

Theorem 5.1.1. *Let M be a manifold. There is a one-to-one correspondence between cosymplectic structures on M and regular corank-one Poisson structures on M, endowed with a transverse Poisson vector field.*

Proof. First assume we are given a pair $(\Pi, X) \in \mathfrak{X}^2(M) \times \mathfrak{X}(M)$, where Π is a Poisson bivector and X is a Poisson vector field transverse to the leaves. We will construct a cosymplectic structure $(\alpha, \omega) \in \Omega^1(M) \times \Omega^2(M)$ that is uniquely defined by the following rules:

$$\begin{cases} \operatorname{Ker}(\alpha_p) = T_pL \quad \text{for all } p \in M, \text{ where } L \text{ is the symplectic leaf through } p. \\ \alpha(X) = 1. \\ \omega|_{\operatorname{Ker}(\alpha)} = -\left(\Pi|_{\operatorname{Ker}(\alpha)}\right)^{-1}. \\ \iota_X \omega = 0. \end{cases} \tag{5.1}$$

Let us first construct $\alpha \in \Omega^1(M)$. The vector field X is a global trivialisation of the normal bundle $TM/T\mathcal{F}$, where $\mathcal{F}$ is the symplectic foliation of Π. Hence also the conormal bundle $(TM/T\mathcal{F})^* = \operatorname{Ann}(T\mathcal{F})$ is trivial, so it has a global trivialisation $\beta \in \Omega^1(M)$. Note that $\operatorname{Ker}(\beta_p) = T_pL$ for all $p \in M$, where L is the leaf through p. Now $\beta(X) = f$, for some $f \in C^\infty(M)$ that is nowhere vanishing. Defining $\alpha := (1/f)\beta$, we have $\operatorname{Ker}(\alpha_p) = T_pL$ for all

$p \in M$, where L is the symplectic leaf through p. And $\alpha(X) = 1$, so α is as desired in (5.1). Next, we construct $\omega \in \Omega^2(M)$. It is well-known that the complex $\Omega^\bullet(\mathcal{F})$ of differential forms along the leaves of $\mathcal{F}$ fits in a short exact sequence of complexes

$$0 \longrightarrow \Omega^\bullet_{\mathcal{F}}(M) \longrightarrow \Omega^\bullet(M) \xrightarrow{r} \Omega^\bullet(\mathcal{F}) \longrightarrow 0, \tag{5.2}$$

where $\Omega^\bullet_{\mathcal{F}}(M)$ is the kernel of the map r, which restricts differential forms on TM to $T\mathcal{F}$. The family of symplectic forms on the leaves of $\mathcal{F}$ defines a foliated differential form $\omega_{\mathcal{F}} \in \Omega^2(\mathcal{F})$, and exactness of the sequence (5.2) implies in particular that we can find $\eta \in \Omega^2(M)$ such that η extends $\omega_{\mathcal{F}}$, i.e. $r(\eta) = \omega_{\mathcal{F}}$. Now $\iota_X\eta$ is a one-form, which we call $\beta \in \Omega^1(M)$. Define $\omega := \eta + \beta \wedge \alpha$, where α is as constructed before. Since $\alpha \in \Gamma(\mathrm{Ann}(T\mathcal{F}))$, we have $r(\omega) = r(\eta) = \omega_{\mathcal{F}}$. Moreover, we have

$$\iota_X\omega = \iota_X\eta + (\iota_X\beta)\,\alpha - \beta\,(\iota_X\alpha) = \iota_X\eta - \beta = 0,$$

since $\iota_X\alpha = 1$ and $\iota_X\beta = \eta(X,X) = 0$ by skew-symmetry. Hence ω satisfies

$$\begin{cases} \omega|_{\mathrm{Ker}(\alpha)} = -\left(\Pi|_{\mathrm{Ker}(\alpha)}\right)^{-1} \\ \iota_X\omega = 0 \end{cases},$$

as required in (5.1). We now show that the pair (α, ω) is a cosymplectic structure on M, following the characterization (4.17). We have by construction that α is nowhere vanishing and that $\omega|_{\mathrm{Ker}(\alpha)}$ is non-degenerate, hence we only have to show that α and ω are closed. To show that $d\alpha = 0$, we only have to check that $d\alpha$ vanishes on pairs of the form (X_g, X_h) and (X, X_g), for $g, h \in C^\infty(M)$. We have

$$\begin{aligned} d\alpha(X_g, X_h) &= X_g(\alpha(X_h)) - X_h(\alpha(X_g)) - \alpha([X_g, X_h]) \\ &= X_g(\alpha(X_h)) - X_h(\alpha(X_g)) - \alpha\left(X_{\{g,h\}}\right) \\ &= 0, \end{aligned}$$

using Lemma 2.7.3 in the second equality. The last equality holds since Hamiltonian vector fields are tangent to the leaves of $\mathcal{F}$ and $\alpha \in \Gamma(\mathrm{Ann}(T\mathcal{F}))$. Next, we have

$$d\alpha(X, X_g) = X(\alpha(X_g)) - X_g(\alpha(X)) - \alpha([X, X_g]). \tag{5.3}$$

As before, $\alpha(X_g) = 1$, and also $X_g(\alpha(X)) = 0$ since $\alpha(X) = 1$. To inspect the last term in (5.3), we compute for $h \in C^\infty(M)$:

$$\begin{aligned} [X, X_g](h) &= X(X_g(h)) - X_g(X(h)) \\ &= X(\{g,h\}) - \{g, X(h)\} \\ &= \{X(g), h\} + \{g, X(h)\} - \{g, X(h)\} \\ &= \{X(g), h\} \\ &= X_{X(g)}(h), \end{aligned}$$

using that X is a Poisson vector field. Hence $[X, X_g] = X_{X(g)}$ and in particular $[X, X_g]$ is tangent to the leaves of $\mathcal{F}$, so that $\alpha([X, X_g]) = 0$. So the right hand side in (5.3) is zero, and we conclude that α is closed. Similarly, to show that ω is closed, we only have to check that $d\omega$ vanishes on triples of the form (X_g, X_h, X_k) and (X, X_g, X_h) for $g, h, k \in C^\infty(M)$. First of all, we have

$$d\omega(X_g, X_h, X_k) = d_{\mathcal{F}}\omega_{\mathcal{F}}(X_g, X_h, X_k) = 0,$$

using that $\omega_{\mathcal{F}}$ is a foliated two-form that is closed for the leafwise de Rham differential $d_{\mathcal{F}}$. Next, we note that

$$\begin{aligned}\omega_p\left(X_g(p),X_h(p)\right) &= \left(\omega_L\right)_p\left(X_g(p),X_h(p)\right)\\ &= \left(\omega_L^\flat\right)_p\left(\Pi_p^\sharp(d_pg)\right)(X_h(p))\\ &= \left(\omega_L^\flat\right)_p\left((\Pi_L)_p^\sharp(d_p(g|_L))\right)(X_h(p))\\ &= -(d_p(g|_L))(X_h(p))\\ &= -(d_pg)(X_h(p))\\ &= -\left(X_h(g)\right)(p)\\ &= \{g,h\}(p),\end{aligned}$$

where L is the leaf through p, ω_L is the symplectic form on L and Π_L is the non-degenerate Poisson structure induced on L. This computation shows that $\omega(X_g,X_h)=\{g,h\}$. Therefore,

$$\begin{aligned}d\omega(X,X_g,X_h) &= X(\omega(X_g,X_h))-X_g(\omega(X,X_h))+X_h(\omega(X,X_g))\\ &\quad-\omega\left([X,X_g],X_h\right)+\omega\left([X,X_h],X_g\right)-\omega\left([X_g,X_h],X\right)\\ &= X(\{g,h\})-X_g\left(\iota_X\omega(X_h)\right)+X_h\left(\iota_X\omega(X_g)\right)\\ &\quad-\omega\left(X_{X(g)},X_h\right)+\omega\left(X_{X(h)},X_g\right)+\iota_X\omega(X_{\{g,h\}})\\ &= X(\{g,h\})-\{X(g),h\}-\{g,X(h)\}\\ &= 0,\end{aligned}$$

where we used that $\iota_X\omega=0$ and that X is a Poisson vector field. Hence also ω is closed, and we conclude that $(\alpha,\omega)\in\Omega^1(M)\times\Omega^2(M)$ is a cosymplectic structure.

For the converse, we start with a cosymplectic structure $(\alpha,\omega)\in\Omega^1(M)\times\Omega^2(M)$. These data determine a codimension-one symplectic foliation $(\mathcal{F},\omega_{\mathcal{F}})$ on M, as follows:

- The one-form $\alpha\in\Omega^1(M)$ is nowhere vanishing and closed, hence $\mathrm{Ker}(\alpha)$ is an involutive corank-one distribution. Indeed, if $X,Y\in\Gamma(\mathrm{Ker}(\alpha))$ then

$$0=d\alpha(X,Y)=X(\alpha(Y))-Y(\alpha(X))-\alpha([X,Y])=-\alpha([X,Y]),$$

  so that $[X,Y]\in\Gamma(\mathrm{Ker}(\alpha))$. Frobenius' theorem gives a uniquely determined codimension-one foliation $\mathcal{F}$ of M integrating the distribution $\mathrm{Ker}(\alpha)$.

- By the characterization (4.17), we know that $\omega|_{\mathrm{Ker}(\alpha)}$ is non-degenerate, hence ω pulls back to a symplectic form on each leaf of $\mathcal{F}$. Therefore, ω defines a leafwise symplectic form $\omega_{\mathcal{F}}$.

In [Vai], one shows that there exists a unique Poisson structure Π on M inducing the given symplectic foliation $(\mathcal{F},\omega_{\mathcal{F}})$. Namely, we can define a Poisson bracket on $C^\infty(M)$ by

$$\{f,g\}(x):=\omega_L\left(X_f^L,X_g^L\right)(x),$$

where L is the leaf passing through x and $X_f^L,X_g^L\in\mathfrak{X}(L)$ are the Hamiltonian vector fields computed with the symplectic structure ω_L on L. Evidently, Π is of corank one.
Next, we attach to the pair (α,ω) a vector field $X\in\mathfrak{X}(M)$, uniquely defined by the rules

$$\begin{cases}\iota_X\omega=0\\ \alpha(X)=1\end{cases}. \tag{5.4}$$

To construct this vector field, we proceed as follows. The one-form $\alpha \in \Omega^1(M)$ trivializes the conormal bundle $(TM/T\mathcal{F})^*$, hence also the normal bundle $TM/T\mathcal{F}$ is trivial. So we can choose $Y \in \mathfrak{X}(M)$ such that Y is nowhere tangent to the leaves of $\mathcal{F}$. Rescaling Y, we can make sure that $\alpha(Y) = 1$. We now have that $\iota_Y\omega$ is a one-form on M, which we call β. Define $X := Y + \Pi^\sharp(\beta)$. Since $\Pi^\sharp(\beta)$ is tangent to the leaves of $\mathcal{F}$, and $\alpha \in \Gamma(\mathrm{Ann}(T\mathcal{F}))$, we still have $\alpha(X) = \alpha(Y) = 1$. But we also claim that $\iota_X\omega = 0$. Indeed, we will show that

$$\iota_{\Pi_p^\sharp(\beta_p)}\omega_p = -\beta_p$$

and to do this, we have to show equality on the vectors Y_p and V_p, where V_p is an arbitrary vector in T_pL and L is the leaf through p. We have

$$\left(\iota_{\Pi_p^\sharp(\beta_p)}\omega_p\right)(Y_p) = -\left(\iota_{Y_p}\omega_p\right)\left(\Pi_p^\sharp(\beta_p)\right) = -\beta_p\left(\Pi_p^\sharp(\beta_p)\right) = -\Pi_p(\beta_p, \beta_p) = 0,$$

and also

$$-\beta_p(Y_p) = -\left(\iota_{Y_p}\omega_p\right)(Y_p) = -\omega_p(Y_p, Y_p) = 0.$$

Next, using that $L \subset M$ is a Poisson submanifold with induced non-degenerate Poisson structure Π_L satisfying $\Pi_L = -\omega_L^{-1}$, we have

$$\begin{aligned}
\left(\iota_{\Pi_p^\sharp(\beta_p)}\omega_p\right)(V_p) &= \omega_p\left(\Pi_p^\sharp(\beta_p), V_p\right) \\
&= \left(\omega_L^\flat\right)_p\left(\Pi_p^\sharp(\beta_p)\right)(V_p) \\
&= \left(\omega_L^\flat\right)_p\left((\Pi_L^\sharp)_p(\widetilde{\beta}_p)\right)(V_p) \\
&= -\widetilde{\beta}_p(V_p) \\
&= -\beta_p(V_p),
\end{aligned}$$

where we denoted by $\widetilde{\beta}$ the pullback of β to the leaf L. So we showed that $\iota_X\omega = 0$, and therefore $X \in \mathfrak{X}(M)$ is as desired in (5.4). For sure, X is transverse to the leaves of Π since $\alpha(X) = 1$ is nowhere zero. It remains to show that X is Poisson, i.e. that $\pounds_X\Pi = 0$. Since Π is constructed out of α and ω, it suffices to show that $\pounds_X\alpha = \pounds_X\omega = 0$. This is readily checked, since by Cartan's magic formula

$$\pounds_X\alpha = d(\iota_X\alpha) + \iota_X d\alpha = d(1) + 0 = 0$$

and

$$\pounds_X\omega = d(\iota_X\omega) + \iota_X d\omega = 0,$$

using that $\alpha(X) = 1$ and $\iota_X\omega = 0$, along with the fact that α and ω are closed. So the pair $(\Pi, X) \in \mathfrak{X}^2(M) \times \mathfrak{X}(M)$ indeed consists of a corank-one Poisson structure and a transverse Poisson vector field. At last, the assignments described above are clearly inverse to each other, and therefore the theorem is proved. □

So the duality between two-forms and bivectors is rather subtle. In the non-degenerate case, there is a one-to-one correspondence between non-degenerate Poisson bivectors and symplectic forms. When the rank is not maximal, the duality is less straightforward. A cosymplectic pair $(\alpha, \omega) \in \Omega^1(M) \times \Omega^2(M)$ induces a corank-one Poisson structure Π on M; there are however several such pairs inducing Π. The ambiguity disappears once we specify a direction transverse to the leaves.

With Theorem 5.1.1 in mind, we obtain the following addendum to Lemma 4.2.5.

Lemma 5.1.2. *Let ω be a b-symplectic form on (M, Z), and let Π be the dual log-symplectic structure. As in Lemma 4.2.5, we can decompose*

$$\omega = \alpha + d\log(\lambda) \wedge p^*(\theta), \tag{5.5}$$

for some choice of distance function λ. Here $\theta \in \Omega^1(Z)$ and $\alpha \in \Omega^2(M)$ are closed, and the map $p : E \to Z$ is the projection in a tubular neighborhood of Z. Let $i : Z \hookrightarrow M$ denote the inclusion and set $\widetilde{\alpha} := i^*\alpha$. Then $(\theta, \widetilde{\alpha})$ is the cosymplectic structure on Z corresponding with the pair $\big(\Pi|_Z, \Pi^\sharp(d\log(\lambda))|_Z\big)$.

Proof. For short, we write $\Pi_Z := \Pi|_Z$ and

$$X := \Pi^\sharp(d\log(\lambda)) = -(\omega^\flat)^{-1}(d\log(\lambda)),$$

so that $\omega^\flat(X) = -d\log(\lambda)$. We have to check that the conditions (5.1) hold, i.e. that

$$\begin{cases} \mathrm{Ker}(\theta_p) = T_pL \quad \text{for all } p \in Z, \text{ where } L \text{ is the symplectic leaf of } \Pi_Z \text{ through } p. \\ \theta(X|_Z) = 1. \\ \widetilde{\alpha}|_{\mathrm{Ker}(\theta)} = -\left(\Pi_Z|_{\mathrm{Ker}(\theta)}\right)^{-1}. \\ \iota_{X|_Z}\widetilde{\alpha} = 0. \end{cases} \tag{5.6}$$

By Remark 4.2.11, we already know that the first and third condition in (5.6) are satisfied. In Theorem 4.3.7, it is shown that $\Pi^\sharp \circ d = -d_\Pi \circ \Pi^\sharp$, which implies that $\Pi^\sharp$ takes closed b-one-forms to Poisson b-vector fields. Hence, X is Poisson and tangent to Z, which implies that $X|_Z$ is Poisson for Π_Z since (Z, Π_Z) is a Poisson submanifold of (M, Π). Restricting (5.5) to Z and contracting with $X|_Z$ gives

$$\iota_{X|_Z}\widetilde{\alpha} + \iota_{X|_Z} d\log(\lambda)|_Z\theta - d\log(\lambda)|_Z\theta(X|_Z) = -d\log(\lambda)|_Z,$$

hence

$$\iota_{X|_Z}\tilde{\alpha} + \iota_{X|_Z} d\log(\lambda)|_Z\theta = d\log(\lambda)|_Z(\theta(X|_Z) - 1). \tag{5.7}$$

Recall that at $p \in Z$, we have the direct sum decomposition

$${}^bT_p^*M = T_p^*Z \oplus \langle (d\log(\lambda))_p \rangle.$$

Since at each point $p \in Z$, the left hand side of (5.7) lives in T_p^*Z and the right hand side lives in $\langle (d\log(\lambda))_p \rangle$, both must be zero. Since $d\log(\lambda)|_Z$ is non-vanishing, we hence have

$$\theta(X|_Z) = 1 \qquad \text{and} \qquad \iota_{X|_Z}\tilde{\alpha} + \iota_{X|_Z} d\log(\lambda)|_Z\theta = 0.$$

Now

$$\begin{aligned} \iota_{X|_Z} d\log(\lambda)|_Z &= \langle d\log(\lambda)|_Z, X|_Z \rangle = \langle d\log(\lambda)|_Z, \Pi^\sharp(d\log(\lambda))|_Z \rangle \\ &= \langle d\log(\lambda)|_Z, (\Pi_Z)^\sharp(d\log(\lambda)|_Z) \rangle = \Pi_Z\left(d\log(\lambda)|_Z, d\log(\lambda)|_Z\right) = 0 \end{aligned}$$

by skew-symmetry. Hence

$$\theta(X|_Z) = 1 \qquad \text{and} \qquad \iota_{X|_Z}\tilde{\alpha} = 0.$$

This shows that $X|_Z$ is transverse to the leaves of Z, and that the second and fourth condition in (5.6) are satisfied. ☐

5.2 Normal form

We now present a semilocal normal form for orientable log-symplectic structures, which is valid on a neighborhood of the singular locus. The theorem appeared in [BOT, Section 4.1].

Theorem 5.2.1. *Let Π be a log-symplectic structure on an orientable manifold M^{2n}, with singular locus $Z \neq \emptyset$. Let X be a modular vector field on M, for some choice of volume form. We then have:*

i) $\Pi_Z := \Pi|_Z$ *is a regular corank-one Poisson structure on Z and $X_Z := X|_Z$ is a transverse Poisson vector field. Moreover, there is a tubular neighborhood $O \subset Z \times \mathbb{R}$ of Z, in which Z corresponds to $t = 0$, such that*

$$\Pi|_O = X_Z \wedge t\frac{\partial}{\partial t} + \Pi_Z. \tag{5.8}$$

ii) Let $(\theta, \eta) \in \Omega^1(Z) \times \Omega^2(Z)$ be the cosymplectic structure corresponding to $(\Pi_Z, -X_Z)$. Then the b-symplectic form ω dual to Π can be written as

$$\omega|_O = \frac{dt}{t} \wedge \theta + \eta.$$

We already know that the first sentence of statement *i*) is true. Indeed, Π_Z is a corank-one Poisson structure on Z by Corollary 3.2.3, and we know that X is a Poisson vector field that is tangent to Z and transverse to the leaves inside Z by Proposition 4.2.15. The fact that X_Z is a Poisson vector field on (Z, Π_Z) is merely a consequence of (Z, Π_Z) being a Poisson submanifold of (M, Π). Indeed, let $i : Z \hookrightarrow M$ denote the inclusion. Since Π and X are tangent to Z, we have that Π_Z and Π are i-related, and that X_Z and X are i-related. By Lemma 8.3.2, also $[X_Z, \Pi_Z]$ and $[X, \Pi]$ are i-related. In particular, $[X, \Pi]$ is tangent to Z. Since $d_p i : \wedge^2 T_pZ \to \wedge^2 T_pM$ is injective for all $p \in Z$, there exists a unique bivector on Z that is i-related with $[X, \Pi]$. Since both $[X_Z, \Pi_Z]$ and $[X, \Pi]|_Z$ are i-related with $[X, \Pi]$, we must have that $[X_Z, \Pi_Z] = [X, \Pi]|_Z$. Hence

$$\pounds_{X_Z}\Pi_Z = [X_Z, \Pi_Z] = [X, \Pi]|_Z = (\pounds_X \Pi)|_Z = 0,$$

since X is a Poisson vector field on (M, Π). We now start the actual proof of Theorem 5.2.1.

Proof. (*of Theorem 5.2.1*)

Step 1

We start by constructing a convenient tubular neighborhood U of Z.

Let μ be a volume form on M, and let ξ be its dual $2n$-vector field. Since $\wedge^{2n}TM$ is a line bundle, we have $\Pi^n = t\xi$ for some $t \in C^\infty(M)$ that is a defining function for Z. Note that

$$\langle \Pi^n, \mu \rangle = \langle t\xi, \mu \rangle = t\langle \xi, \mu \rangle = t,$$

since μ and ξ are duals. Since t vanishes linearly on Z, we have that t is a submersion along Z and that 0 is a regular value of t. Now let $U \subset Z \times \mathbb{R}$ be a tubular neighborhood of Z, where we choose some trivialization of the normal bundle NZ (note indeed that the normal bundle of Z is trivial: M is orientable, and so is Z since it has a volume form by Lemma 4.2.5). Let $r : U \to Z$ denote the projection map in the normal bundle. We claim that the map $(r, t) : U \to Z \times \mathbb{R}$ is a local diffeomorphism around Z. To show this, it suffices to check that its derivative is an isomorphism at points $p \in Z$, by the inverse function theorem.

- We see that $d_p(r,t) = (d_pr, d_pt)$ is surjective, since r is a submersion and t is a submersion along Z.
- By the regular value theorem, we know that $T_pZ = T_p\left(t^{-1}(0)\right) = \mathrm{Ker}(d_pt)$. Hence

$$\mathrm{Ker}\left(d_p(r,t)\right) = \mathrm{Ker}\left(d_pr, d_pt\right) = \mathrm{Ker}(d_pr) \cap \mathrm{Ker}(d_pt) = \mathrm{Ker}(d_pr) \cap T_pZ.$$

  Since $d_pr|_{T_pZ} = \mathrm{Id}_{T_pZ}$, this shows that $d_p(r,t)$ is injective.

Shrinking U if necessary, we obtain that $U \subset Z \times \mathbb{R}$ is the desired tubular neighborhood of Z with global coordinate t in the fibers, such that Z corresponds to $t = 0$.[1]

Step 2

In this neighborhood U, we can write

$$\Pi|_U = Y_t \wedge \frac{\partial}{\partial t} + w_t,$$

with $dt(Y_t) = w_t^\sharp(dt) = 0$. Since Π is log-symplectic, we have that Π is tangent to Z and that Π^n vanishes linearly on Z. Therefore, necessarily $Y_t = tV_t$ for some vector field V_t on U satisfying $dt(V_t) = 0$. Hence

$$\Pi|_U = V_t \wedge t\frac{\partial}{\partial t} + w_t.$$

Denote by X the modular vector field corresponding to the volume form μ.

<u>Claim:</u> $V_t = X|_U$.

Choose $f \in C^\infty(U)$. We compute

$$\begin{aligned}
\Pi|_U(df, dt) &= \left(V_t \wedge t\frac{\partial}{\partial t} + w_t\right)(df, dt) \\
&= \left(V_t \wedge t\frac{\partial}{\partial t}\right)(df, dt) \\
&= \begin{vmatrix} df(V_t) & df\left(t\frac{\partial}{\partial t}\right) \\ dt(V_t) & dt\left(t\frac{\partial}{\partial t}\right) \end{vmatrix} \\
&= \begin{vmatrix} df(V_t) & df\left(t\frac{\partial}{\partial t}\right) \\ 0 & t \end{vmatrix} \\
&= tV_t(f).
\end{aligned} \tag{5.9}$$

On the other hand,

$$\{f, t\} = X_f(t) = \pounds_{X_f} t = \pounds_{X_f}\langle \Pi^n, \mu\rangle = \langle \pounds_{X_f}\Pi^n, \mu\rangle + \langle \Pi^n, \pounds_{X_f}\mu\rangle. \tag{5.10}$$

Here, we note:

- Since Hamiltonian vector fields are Poisson, we have $\pounds_{X_f}\Pi = 0$. Therefore also $\pounds_{X_f}\Pi^n = 0$ by induction, using that $\pounds_{X_f}$ is a derivation of the wedge product.

[1] To justify that U is as we want, we can argue as follows. For local coordinates $(V, x_1, \ldots, x_{2n-1})$ on Z, we want $(r^*(x_1), \ldots, r^*(x_{2n-1}), t)$ to be coordinates on $r^{-1}(V) \subset U$. That is, we want

$$(r^*(x_1), \ldots, r^*(x_{2n-1}), t) = (x_1 \circ r, \ldots, x_{2n-1} \circ r, t)$$

to be a diffeomorphism into $\mathbb{R}^{2n}$. This is the case, since it is a composition of the diffeomorphisms (r,t) and $(x_1, \ldots, x_{2n-1}, \mathrm{Id})$.

- By definition of the modular vector field X, we have

$$\pounds_{X_f}\mu = X(f)\mu = (\pounds_X f)\mu.$$

Hence, putting together (5.9) and (5.10), we get

$$tV_t(f) = \Pi|_U(df, dt) = \{f, t\} = \langle \Pi^n, (\pounds_X f)\mu \rangle = (\pounds_X f)t = X(f)t.$$

This implies that $V_t(f) = X(f)$ on the locus $(U \setminus Z) \leftrightarrow \{t \neq 0\}$, so that by continuity $V_t(f) = X(f)$ on all of U. This proves the claim that $V_t = X|_U$. In conclusion, we have

$$\Pi|_U = V_t \wedge t\frac{\partial}{\partial t} + w_t,$$

where $w_0 = \Pi|_Z = \Pi_Z$ and $V_0 = X|_Z = X_Z$.

Step 3

Since Π_Z is a corank-one Poisson bivector and X_Z is a Poisson vector field transverse to the leaves of Z, we get a bivector Π_0 on U defined by

$$\Pi_0 = X_Z \wedge t\frac{\partial}{\partial t} + \Pi_Z,$$

which is seen to be log-symplectic by the same reasoning as in Example 3.2.6. (Note that here we consider X_Z and Π_Z as being defined on U by taking their horizontal lifts). The log-symplectic structures $\Pi|_U$ and Π_0 define non-degenerate b-bivector fields, which can be inverted. Let $\omega = -\Pi|_U^{-1}$ and $\omega_0 = -\Pi_0^{-1}$ denote the b-symplectic forms on U that are inverse to $\Pi|_U$ and Π_0, respectively.

<u>Claim:</u> $\omega|_Z = \omega_0|_Z$.

We can decompose

$$\omega_0 = \frac{dt}{t} \wedge \alpha + \beta,$$

where $\alpha \in \Omega^1(Z)$ and $\beta \in \Omega^2(Z)$ are independent of t (we write α and β for short instead of $r^*(\alpha)$ and $r^*(\beta)$, where $r : U \to Z$ is the projection). By Lemma 5.1.2, we have that (α, β) is the cosymplectic structure corresponding with the pair

$$\left(\Pi_0|_Z, \Pi_0^\sharp \left(\frac{dt}{t} \right) \Big|_Z \right) = (\Pi_Z, -X_Z),$$

so that in fact

$$\omega_0 = \frac{dt}{t} \wedge \theta + \eta.$$

Next, we have the equality of b-bivector fields

$$\Pi|_Z = V_0 \wedge \left(t\frac{\partial}{\partial t} \right)\Big|_Z + w_0 = X_Z \wedge \left(t\frac{\partial}{\partial t} \right)\Big|_Z + \Pi_Z = \Pi_0|_Z \in \Gamma(\wedge^2({}^bTM)|_Z),$$

and since inverting is a pointwise operation, this implies that

$$\omega|_Z = \omega_0|_Z = \frac{dt}{t}\Big|_Z \wedge \theta + \eta. \tag{5.11}$$

Step 4

By virtue of the equation (5.11), we can apply the local b-Moser theorem 4.2.7, which gives a diffeomorphism $\phi : O_0 \to O_1$ between neighborhoods of Z such that $\phi|_Z = \mathrm{Id}_Z$ and $\phi^*\omega = \omega_0$. By functoriality, ϕ should push forward Π_0 to Π: let us check this explicitly. For vector fields X, Y we have

$$\begin{aligned}
\left(\phi^*\omega\right)^\flat (X)(Y) &= \left(\phi^*\omega\right)(X,Y) = \omega(\phi_*(X), \phi_*(Y)) = \omega^\flat(\phi_*(X))(\phi_*(Y)) \\
&= \left(\omega^\flat \circ \phi_*\right)(X)(\phi_*(Y)) = \phi^*\left(\left(\omega^\flat \circ \phi_*\right)(X)\right)(Y) \\
&= \left(\phi^* \circ \omega^\flat \circ \phi_*\right)(X)(Y),
\end{aligned}$$

so that $\left(\phi^*\omega\right)^\flat = \phi^* \circ \omega^\flat \circ \phi_*$. Then we have for one-forms α_1, α_2 that

$$\begin{aligned}
(\phi_*\Pi_0)(\alpha_1,\alpha_2) &= \Pi_0(\phi^*\alpha_1, \phi^*\alpha_2) = \left\langle \Pi_0^\sharp(\phi^*\alpha_1), \phi^*\alpha_2 \right\rangle \\
&= -\left\langle \left(\omega_0^\flat\right)^{-1}(\phi^*\alpha_1), \phi^*\alpha_2 \right\rangle \\
&= -\left\langle \left(\left(\phi^*\omega\right)^\flat\right)^{-1}(\phi^*\alpha_1), \phi^*\alpha_2 \right\rangle \\
&= -\left\langle \left((\phi_*)^{-1} \circ \left(\omega^\flat\right)^{-1} \circ (\phi^*)^{-1}\right)(\phi^*\alpha_1), \phi^*\alpha_2 \right\rangle \\
&= -\left\langle \left((\phi_*)^{-1} \circ \left(\omega^\flat\right)^{-1}\right)(\alpha_1), \phi^*\alpha_2 \right\rangle \\
&= -\left\langle (\phi^{-1})_*\left(\left(\omega^\flat\right)^{-1}(\alpha_1)\right), \phi^*\alpha_2 \right\rangle \\
&= -\left\langle \left(\omega^\flat\right)^{-1}(\alpha_1), \left(\phi^{-1}\right)^*(\phi^*\alpha_2) \right) \\
&= -\left\langle \left(\omega^\flat\right)^{-1}(\alpha_1), \alpha_2 \right\rangle \\
&= \left\langle \Pi^\sharp(\alpha_1), \alpha_2 \right\rangle \\
&= \Pi(\alpha_1, \alpha_2).
\end{aligned}$$

So indeed $\phi_*\Pi_0 = \Pi$. Now note that

$$\begin{aligned}
\phi_*\left(\Pi_0\right) &= \phi_*\left(X_Z\right) \wedge \phi_*\left(t\frac{\partial}{\partial t}\right) + \phi_*(\Pi_Z) \\
&= \left(\phi|_Z\right)_*\left(X_Z\right) \wedge \phi_*\left(t\frac{\partial}{\partial t}\right) + \left(\phi|_Z\right)_*\left(\Pi_Z\right) \\
&= X_Z \wedge t\frac{\partial}{\partial t} + \Pi_Z,
\end{aligned}$$

where we use Lemma 5.3.5 in the last equality to see that ϕ_* preserves the normal b-vector field. Hence, setting $O := O_1$ yields the conclusion $i)$ of the theorem, namely

$$\Pi|_O = \phi_*\Pi_0 = X_Z \wedge t\frac{\partial}{\partial t} + \Pi_Z.$$

Taking inverses in this equality then also yields conclusion $ii)$ of the theorem:

$$\omega|_O = \frac{dt}{t} \wedge \theta + \eta.$$

□

Remark 5.2.2. We should stress that the expression (5.8) is very much subject to the choice of volume form. Of course, the modular vector field X requires a choice of volume form, but also the coordinate t depends on the chosen volume form μ, in a more disguised way, as $t = \langle \Pi^n, \mu \rangle$.

5.3 Extensions

Up until now, we always started from a given log-symplectic structure (M, Z, Π) and investigated the local picture near the singular locus Z. In particular, we saw that Π induces a corank-one Poisson structure on Z. Conversely, we can ask: given a b-manifold (M, Z) with a corank-one Poisson structure Π_Z on Z, is Π_Z induced by a log-symplectic structure on M? And if so, to what extent are such log-symplectic extensions of Π_Z unique?

5.3.1 Existence of extensions

We give necessary and sufficient conditions for a corank-one Poisson structure Π_Z on Z to be induced by a log-symplectic structure on a tubular neighborhood of Z. We will restrict ourselves to orientable manifolds. This theorem is a combination of [GMP2, Theorem 50] and [GMP1, Proposition 18].

Theorem 5.3.1. *Let (M^{2n}, Z) be a b-manifold, with M and Z orientable, and let Π_Z be a corank-one Poisson structure on Z. The following are equivalent:*

i) There exist a tubular neighborhood U of Z and a log-symplectic structure Π on U that induces Π_Z.

ii) There exists a Poisson vector field on Z that is transverse to the symplectic leaves.

iii) The foliation of Z has a closed defining one-form and a closed two-form that pulls back to the symplectic form on each leaf.

Proof. We will prove that $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow i)$.

- First assume that there exist a tubular neighborhood $U \subset M$ of Z and a log-symplectic structure Π on U inducing Π_Z. Since M is orientable, we can choose a volume form μ on U and consider the modular vector field X associated with μ and Π. Then X is a Poisson vector field on U (see Theorem 2.9.5) that is tangent to Z and transverse to the symplectic leaves of Z (see Proposition 4.2.15). Restricting X to Z gives the desired transverse Poisson vector field on (Z, Π_Z).

- Let X_Z be a Poisson vector field for (Z, Π_Z) that is transverse to the symplectic leaves. The pair (Π_Z, X_Z) determines a cosymplectic structure $(\alpha, \omega) \in \Omega^1(Z) \times \Omega^2(Z)$ on Z by Theorem 5.1.1. Then α is a closed defining one-form for the foliation of Z, and ω is a closed two-form that pulls back to the symplectic form on each leaf, as is clear from (5.1).

- Assume that $\alpha \in \Omega^1(Z)$ is a closed defining one-form for the foliation on Z, and that $\omega \in \Omega^2(Z)$ is a closed two-form that pulls back to the symplectic form on each leaf. Then α is nowhere vanishing, and $\omega|_{\mathrm{Ker}(\alpha)}$ is non-degenerate, which by (4.17) implies that $\alpha \wedge \omega^{n-1}$ is nowhere vanishing. Let $U \subset NZ$ be a tubular neighborhood of Z in the normal bundle NZ, and let $p : U \to Z$ be the projection. By orientability of M and Z, the normal bundle NZ is trivial; let t be a global coordinate in the fibers. We define a b-form $\widetilde{\omega} \in {}^b\Omega^2(U)$ by
$$\widetilde{\omega} = p^*(\omega) + \frac{dt}{t} \wedge p^*(\alpha). \tag{5.12}$$

We claim that $\widetilde{\omega}$ is a b-symplectic form. Clearly, $\widetilde{\omega}$ is closed since

$$d\widetilde{\omega} = d(p^*(\omega)) + \frac{dt}{t} \wedge d(p^*(\alpha)) = p^*(d\omega) + \frac{dt}{t} \wedge p^*(d\alpha) = 0,$$

using that α and ω are closed. To show that $\widetilde{\omega}$ is non-degenerate, we have to check that

$$\widetilde{\omega}^n = n\frac{dt}{t} \wedge p^*(\alpha) \wedge (p^*(\omega))^{n-1}$$

is a nowhere vanishing b-form. Assume by contradiction that $\widetilde{\omega}^n_q = 0$ for some $q \in U$. Then in particular,

$$0 = \left(\iota_{t\frac{\partial}{\partial t}}\widetilde{\omega}^n\right)_q = \left[p^*(\alpha) \wedge (p^*(\omega))^{n-1}\right]_q = \left[p^*(\alpha \wedge \omega^{n-1})\right]_q,$$

and this implies that $\left(\alpha \wedge \omega^{n-1}\right)_{p(q)} = 0$ since $d_qp : T_qU \to T_{p(q)}Z$ is surjective. So we run into a contradiction, and we conclude that $\widetilde{\omega}$ is a b-symplectic form on U. Let $\Pi \in \Gamma(\wedge^2 TU)$ be its dual log-symplectic structure. In Remark 4.2.11, we showed that the symplectic foliation on Z determined by the pair (α, ω) is exactly the symplectic foliation induced by $\Pi|_Z \in \Gamma(\wedge^2 TZ)$. That is, we have corank-one Poisson structures Π_Z and $\Pi|_Z$ on Z inducing the same symplectic foliation of Z. By Proposition 2.12.17, they must coincide: $\Pi_Z = \Pi|_Z$. Hence Π_Z is induced by the log-symplectic structure Π on U, which finishes the proof.

□

Remark 5.3.2. If we are only given the data (Z, Π_Z), then we can thicken Z to $M := Z \times (-\epsilon, \epsilon)$ and construct a log-symplectic structure on M inducing Π_Z as in the last point of above proof, provided that the conditions of Theorem 5.3.1 are satisfied.

Example 5.3.3 ([GMP2])**.** Let $Z = S^3$ and Π_Z any corank-one Poisson structure on Z. Assume that the induced symplectic foliation on Z would have a closed defining one-form $\alpha \in \Omega^1(Z)$. Since $H^1(Z) = 0$, necessarily α is exact: $\alpha = df$ for some $f \in C^\infty(Z)$. Since Z is compact, the function f reaches maximum and minimum values, so that df has zeros on Z. Hence α does vanish at some points, and therefore it cannot define a corank-one foliation. By contradiction, we conclude that S^3 cannot be the singular hypersurface of a log-symplectic manifold.

Example 5.3.4 ([GMP2])**.** Consider $Z = \mathbb{T}^3$ with coordinates $\theta_1, \theta_2, \theta_3$. Let $a, b \in \mathbb{R}$ be fixed constants. The map

$$f : \mathbb{T}^3 \to \mathbb{R} : (\theta_1, \theta_2, \theta_3) \mapsto \theta_3 - a\theta_1 - b\theta_2$$

is a submersion, and therefore it gives rise to a codimension-one foliation $\mathcal{F}$ on $\mathbb{T}^3$ whose leaves are the different f-fibers

$$f^{-1}(k) = \{(\theta_1, \theta_2, \theta_3) : \theta_3 = a\theta_1 + b\theta_2 + k\}, \qquad k \in \mathrm{Im}(f) \subset \mathbb{R}.$$

The foliation $\mathcal{F}$ is defined by the one-form $\alpha \in \Omega^1(\mathbb{T}^3)$, given by

$$\alpha = \frac{a}{a^2+b^2+1}d\theta_1 + \frac{b}{a^2+b^2+1}d\theta_2 - \frac{1}{a^2+b^2+1}d\theta_3.$$

Indeed, clearly α is non-vanishing, and for any leaf $L = f^{-1}(k)$ we have

$$i_L^*\alpha = \frac{a}{a^2+b^2+1}d\theta_1 + \frac{b}{a^2+b^2+1}d\theta_2 - \frac{1}{a^2+b^2+1}d(a\theta_1 + b\theta_2 + k)$$

$$
\begin{aligned}
&= \frac{a}{a^2+b^2+1} d\theta_1 + \frac{b}{a^2+b^2+1} d\theta_2 - \frac{a}{a^2+b^2+1} d\theta_1 - \frac{b}{a^2+b^2+1} d\theta_2 \\
&= 0,
\end{aligned}
$$

where $i_L : L \hookrightarrow \mathbb{T}^3$ is the inclusion. We will now construct a Poisson structure Π_Z on Z which induces the foliation $\mathcal{F}$, and which endows each leaf L with a symplectic form that is the pullback to L of

$$
\omega = d\theta_1 \wedge d\theta_2 + b d\theta_1 \wedge d\theta_3 - a d\theta_2 \wedge d\theta_3.
$$

Note that

$$
\begin{aligned}
i_L^*\omega &= d\theta_1 \wedge d\theta_2 + b d\theta_1 \wedge (a d\theta_1 + b d\theta_2) - a d\theta_2 \wedge (a d\theta_1 + b d\theta_2) \\
&= (1+a^2+b^2) d\theta_1 \wedge d\theta_2.
\end{aligned}
$$

Hence necessarily

$$
\Pi_Z|_L = -\left(i_L^*\omega\right)^{-1} = \left(\frac{1}{1+a^2+b^2}\right) \frac{\partial}{\partial\theta_1} \wedge \frac{\partial}{\partial\theta_2}
$$

Since $(L, \Pi_Z|_L)$ has to be a Poisson submanifold of (Z, Π_Z), we must have

$$
\begin{aligned}
(\Pi_Z)_p (d_p\theta_1, d_p\theta_2) &= (\Pi_Z|_L)_p (d_p\theta_1, d_p\theta_2) = \frac{1}{1+a^2+b^2}, \\
(\Pi_Z)_p (d_p\theta_1, d_p\theta_3) &= (\Pi_Z|_L)_p (d_p\theta_1, a d_p\theta_1 + b d_p\theta_2) = \frac{b}{1+a^2+b^2}, \\
(\Pi_Z)_p (d_p\theta_2, d_p\theta_3) &= (\Pi_Z|_L)_p (d_p\theta_2, a d_p\theta_1 + b d_p\theta_2) = -\frac{a}{1+a^2+b^2},
\end{aligned}
$$

where L is the leaf through p. This shows that we should define

$$
\Pi_Z := \left(\frac{1}{1+a^2+b^2}\right) \frac{\partial}{\partial\theta_1} \wedge \frac{\partial}{\partial\theta_2} + \left(\frac{b}{1+a^2+b^2}\right) \frac{\partial}{\partial\theta_1} \wedge \frac{\partial}{\partial\theta_3} - \left(\frac{a}{1+a^2+b^2}\right) \frac{\partial}{\partial\theta_2} \wedge \frac{\partial}{\partial\theta_3}.
$$

One easily sees that Π_Z is Poisson, i.e. that $[\Pi_Z, \Pi_Z] = 0$. Indeed, since the coefficients of Π_Z are constant, the derivation property of $[\cdot,\cdot]$ with respect to the wedge product reduces $[\Pi_Z, \Pi_Z]$ to Lie brackets of coordinate vector fields, which are all zero. It remains to check that Π_Z indeed induces the foliation $\mathcal{F}$. We have

$$
\begin{aligned}
\mathrm{Im}(\Pi_Z^\sharp)_p &= \mathrm{span}\left\{ \left(\Pi_Z^\sharp\right)_p (d_p\theta_1), \left(\Pi_Z^\sharp\right)_p (d_p\theta_2), \left(\Pi_Z^\sharp\right)_p (d_p\theta_3) \right\} \\
&= \mathrm{span}\Bigg\{ \left(\frac{1}{1+a^2+b^2}\right) \frac{\partial}{\partial\theta_2}\bigg|_p + \left(\frac{b}{1+a^2+b^2}\right) \frac{\partial}{\partial\theta_3}\bigg|_p, \\
&\qquad - \left(\frac{1}{1+a^2+b^2}\right) \frac{\partial}{\partial\theta_1}\bigg|_p - \left(\frac{a}{1+a^2+b^2}\right) \frac{\partial}{\partial\theta_3}\bigg|_p, \\
&\qquad - \left(\frac{b}{1+a^2+b^2}\right) \frac{\partial}{\partial\theta_1}\bigg|_p + \left(\frac{a}{1+a^2+b^2}\right) \frac{\partial}{\partial\theta_2}\bigg|_p \Bigg\} \\
&= \mathrm{span}\left\{ \frac{\partial}{\partial\theta_2}\bigg|_p + b\,\frac{\partial}{\partial\theta_3}\bigg|_p, \frac{\partial}{\partial\theta_1}\bigg|_p + a\,\frac{\partial}{\partial\theta_3}\bigg|_p, b\,\frac{\partial}{\partial\theta_1}\bigg|_p - a\,\frac{\partial}{\partial\theta_2}\bigg|_p \right\} \\
&= \mathrm{span}\left\{ \frac{\partial}{\partial\theta_2}\bigg|_p + b\,\frac{\partial}{\partial\theta_3}\bigg|_p, \frac{\partial}{\partial\theta_1}\bigg|_p + a\,\frac{\partial}{\partial\theta_3}\bigg|_p \right\},
\end{aligned}
$$

where the last equality holds since

$$b\,\frac{\partial}{\partial\theta_1}\bigg|_p - a\,\frac{\partial}{\partial\theta_2}\bigg|_p = b\left(\frac{\partial}{\partial\theta_1}\bigg|_p + a\,\frac{\partial}{\partial\theta_3}\bigg|_p\right) - a\left(\frac{\partial}{\partial\theta_2}\bigg|_p + b\,\frac{\partial}{\partial\theta_3}\bigg|_p\right).$$

On the other hand, if $L = f^{-1}(k)$ is the leaf of $\mathcal{F}$ through p, then the preimage theorem gives

$$T_pL = \mathrm{Ker}(d_pf),$$

where

$$d_pf = \left(\frac{\partial f}{\partial\theta_1}(p), \frac{\partial f}{\partial\theta_2}(p), \frac{\partial f}{\partial\theta_3}(p)\right) = (-a, -b, 1).$$

Hence

$$\begin{aligned} T_pL &= \{(x,y,z) \in \mathbb{R}^3 : -ax - by + z = 0\} \\ &= \{(x, y, ax+by) : x, y \in \mathbb{R}\} \\ &= \mathrm{span}\{(1,0,a), (0,1,b)\}, \end{aligned}$$

so that

$$T_pL = \mathrm{span}\left\{\frac{\partial}{\partial\theta_2}\bigg|_p + b\,\frac{\partial}{\partial\theta_3}\bigg|_p, \frac{\partial}{\partial\theta_1}\bigg|_p + a\,\frac{\partial}{\partial\theta_3}\bigg|_p\right\} = \mathrm{Im}(\Pi_Z^\sharp)_p.$$

In conclusion, we have a corank-one Poisson structure Π_Z on Z, whose symplectic foliation has a closed defining one-form α and a closed two-form ω that pulls back to the symplectic form on each leaf. Hence, thickening Z to $M := Z \times (-\epsilon, \epsilon)$, we have that Π_Z is induced by a log-symplectic structure on M. As in the proof of Theorem 5.3.1, a b-symplectic form on M inducing Π_Z is

$$\widetilde{\omega} = p^*(\omega) + \frac{dt}{t} \wedge p^*(\alpha),$$

where t is the coordinate on $(-\epsilon, \epsilon)$ and $p : Z \times (-\epsilon,\epsilon) \to Z$ is the projection. So we can take

$$\widetilde{\omega} = d\theta_1 \wedge d\theta_2 + b d\theta_1 \wedge d\theta_3 - a d\theta_2 \wedge d\theta_3 + \frac{dt}{t} \wedge \left(\frac{a}{a^2+b^2+1}d\theta_1 + \frac{b}{a^2+b^2+1}d\theta_2 - \frac{1}{a^2+b^2+1}d\theta_3\right).$$

Inverting $\widetilde{\omega}$ then gives a log-symplectic structure Π on M inducing Π_Z. We compute

$$-\begin{pmatrix} 0 & 1 & b & -\frac{a}{a^2+b^2+1} \\ -1 & 0 & -a & -\frac{b}{a^2+b^2+1} \\ -b & a & 0 & \frac{1}{a^2+b^2+1} \\ \frac{a}{a^2+b^2+1} & \frac{b}{a^2+b^2+1} & -\frac{1}{a^2+b^2+1} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{a^2+b^2+1} & \frac{b}{a^2+b^2+1} & -a \\ -\frac{1}{a^2+b^2+1} & 0 & -\frac{a}{a^2+b^2+1} & -b \\ -\frac{b}{a^2+b^2+1} & \frac{a}{a^2+b^2+1} & 0 & 1 \\ a & b & -1 & 0 \end{pmatrix},$$

hence

$$\begin{aligned} \Pi &= \left(\frac{1}{a^2+b^2+1}\right)\frac{\partial}{\partial\theta_1} \wedge \frac{\partial}{\partial\theta_2} + \left(\frac{b}{a^2+b^2+1}\right)\frac{\partial}{\partial\theta_1} \wedge \frac{\partial}{\partial\theta_3} - a\frac{\partial}{\partial\theta_1} \wedge \left(t\frac{\partial}{\partial t}\right) \\ &\quad - \left(\frac{a}{a^2+b^2+1}\right)\frac{\partial}{\partial\theta_2} \wedge \frac{\partial}{\partial\theta_3} - b\frac{\partial}{\partial\theta_2} \wedge \left(t\frac{\partial}{\partial t}\right) + \frac{\partial}{\partial\theta_3} \wedge \left(t\frac{\partial}{\partial t}\right) \\ &= \left(t\frac{\partial}{\partial t}\right) \wedge \left(a\frac{\partial}{\partial\theta_1} + b\frac{\partial}{\partial\theta_2} - \frac{\partial}{\partial\theta_3}\right) + \left(\frac{1}{1+a^2+b^2}\right)\frac{\partial}{\partial\theta_1} \wedge \frac{\partial}{\partial\theta_2} + \left(\frac{b}{1+a^2+b^2}\right)\frac{\partial}{\partial\theta_1} \wedge \frac{\partial}{\partial\theta_3} \\ &\quad - \left(\frac{a}{1+a^2+b^2}\right)\frac{\partial}{\partial\theta_2} \wedge \frac{\partial}{\partial\theta_3} \\ &= \left(t\frac{\partial}{\partial t}\right) \wedge \left(a\frac{\partial}{\partial\theta_1} + b\frac{\partial}{\partial\theta_2} - \frac{\partial}{\partial\theta_3}\right) + \Pi_Z. \end{aligned}$$

5.3.2 Equivalence of extensions

Having determined when a corank-one Poisson structure Π_Z allows a log-symplectic extension, we now want to know to what extent such a log-symplectic extension is unique. We will see that, up to a certain notion of equivalence, the log-symplectic extensions of Π_Z, defined in some tubular neighborhood of Z, are parameterized by the cohomology classes in $H^1_{\Pi_Z}(Z)$ of Poisson vector fields transverse to the symplectic leaves.

The material we discuss in this subsection is addressed in [GMP2], but the exposition given there is flawed. Below, we improve on the work done in [GMP2]. As such, while the results in this subsection are not all original, some of the proofs are.

Construction of the correspondence

Let (M, Z) be a b-manifold. We will assume throughout that both M and Z are orientable, so that Z has a defining function that exists on a tubular neighborhood of Z (see Lemma 4.1.2). Recall that we have a canonical short exact sequence of vector bundles

$$0 \to \mathbb{L}_Z \overset{i}{\hookrightarrow} {}^bTM|_Z \overset{\rho|_Z}{\to} TZ \to 0, \tag{5.13}$$

where $\rho|_Z$ is the restriction to Z of the anchor map $\rho : {}^bTM \to TM$ and $\mathbb{L}_Z$ is its kernel. We have seen that $\mathbb{L}_Z$ is a trivial line bundle, with canonical non-vanishing section ξ. This section can be described as $\xi = fv|_Z$ where f is any defining function for Z and v is a vector field with $df(v) = 1$ (see Remark 4.1.12).

Lemma 5.3.5. *Let (M, Z) be a b-manifold with M and Z orientable, and let $\varphi : M \to M$ be a diffeomorphism such that $\varphi|_Z = Id_Z$. Then the b-derivative*[2] *${}^b\varphi_*|_Z : {}^bTM|_Z \to {}^bTM|_Z$ commutes with the maps in* (5.13)*, i.e. we have a commutative diagram*

$$\begin{array}{ccccc} & & {}^bTM|_Z & & \\ & \nearrow & \uparrow & \searrow^{\rho|_Z} & \\ \mathbb{L}_Z & & {}^b\varphi_*|_Z & & TZ \\ & \searrow & & \nearrow_{\rho|_Z} & \\ & & {}^bTM|_Z & & \end{array} .$$

Proof. By definition of the map ${}^b\varphi_*$, we have a commutative diagram

$$\begin{array}{ccc} {}^bTM & \xrightarrow{{}^b\varphi_*} & {}^bTM \\ \downarrow \rho & & \downarrow \rho \\ TM & \xrightarrow{\varphi_*} & TM \end{array} , \tag{5.14}$$

where $\rho : {}^bTM \to TM$ is the anchor map. Over Z, we have

$$\varphi_*|_Z \circ \rho|_Z = (\varphi|_Z)_* \circ \rho|_Z = \rho|_Z,$$

[2] We introduced the b-derivative in previous chapter. There we used the notation ${}^bd\varphi$, but for consistency with our notation for the usual derivative φ_*, we will from now on denote the b-derivative by ${}^b\varphi_*$.

where the first equality holds since $\mathrm{Im}(\rho|_Z)$ is tangent to Z, and the second equality is true since $\varphi|_Z = \mathrm{Id}_Z$. Hence by commutativity of (5.14), we get

$$\rho|_Z \circ {}^b\varphi_*|_Z = \varphi_*|_Z \circ \rho|_Z = \rho|_Z.$$

Next, let f be a defining function for Z and v a vector field such that $df(v) = 1$. Then

$${}^b\varphi_*(fv) = \left(f \circ \varphi^{-1}\right) \varphi_* v,$$

where $f \circ \varphi^{-1}$ is again a defining function for Z, and

$$d(f \circ \varphi^{-1})(\varphi_* v) = df(v) = 1.$$

Hence ${}^b\varphi_*|_Z$ takes the canonical non-vanishing section of $\mathbb{L}_Z$ to itself, which shows that ${}^b\varphi_*|_Z$ is the identity on $\mathbb{L}_Z$. $\square$

Example 5.3.6. Let $(M, Z) = \big((-1,1) \times \mathbb{R}, \{y = 0\}\big)$ and consider the sheer transformation

$$\varphi : M \to M : (x, y) \mapsto (x, y(1 + x)).$$

Then φ is the identity map on Z, and φ is a diffeomorphism since its Jacobian determinant is

$$\begin{vmatrix} 1 & 0 \\ y & 1+x \end{vmatrix} = 1 + x, \tag{5.15}$$

which is nowhere vanishing as $x \in (-1, 1)$. For the b-derivative ${}^b\varphi_* : {}^bTM \to {}^bTM$, we observe that

$${}^b\varphi_*\left(\frac{\partial}{\partial x}\right) = \varphi_*\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

where the last equality holds by considering the first column in the matrix (5.15). Similarly,

$${}^b\varphi_*\left(y\frac{\partial}{\partial y}\right) = \left(y \circ \varphi^{-1}\right)\varphi_*\left(\frac{\partial}{\partial y}\right) = \left(\frac{y}{1+x}\right)(1+x)\frac{\partial}{\partial y} = y\frac{\partial}{\partial y}.$$

Restricting to Z, these results are in perfect agreement with Lemma 5.3.5.

The dual of the short exact sequence (5.13) is

$$0 \to T^*Z \overset{\rho|_Z^*}{\to} {}^bT^*M|_Z \overset{i^*}{\to} \mathbb{L}_Z^* \to 0. \tag{5.16}$$

The bundle $\mathbb{L}_Z^*$ is also trivial, and a trivialization is the dual section ξ^* of ξ. A splitting of the sequence (5.16) is given by the map

$$\psi : \mathbb{L}_Z^* \to {}^bT^*M|_Z : \xi^* \mapsto \left.\frac{df}{f}\right|_Z.$$

Indeed,

$$i^*\left(\left.\frac{df}{f}\right|_Z\right) = \xi^*$$

since

$$\left\langle i^*\left(\left.\frac{df}{f}\right|_Z\right), fv|_Z\right\rangle = \left\langle \left.\frac{df}{f}\right|_Z, i\left(fv|_Z\right)\right\rangle = \left\langle \left.\frac{df}{f}\right|_Z, fv|_Z\right\rangle = 1.$$

Basic commutative algebra says that there exists a (unique) map $\phi : {}^bT^*M|_Z \to T^*Z$ such that[3]

$$\phi \circ \rho|_Z^* = \mathrm{Id} \quad \text{and} \quad \rho|_Z^* \circ \phi + \psi \circ i^* = \mathrm{Id}.$$

We then have

$$\phi \circ \psi \circ i^* = \phi \circ (\mathrm{Id} - \rho|_Z^* \circ \phi) = \phi - \phi = 0,$$

and since i^* is surjective, this implies that $\phi \circ \psi = 0$. Dualizing (5.16) again gives a split exact sequence

$$0 \to \mathbb{L}_Z \underset{\psi^*}{\overset{\mathrm{i}}{\rightleftarrows}} {}^bTM|_Z \underset{\phi^*}{\overset{\rho|_Z}{\rightleftarrows}} TZ \to 0, \tag{5.17}$$

where $\psi^* \circ \phi^* = (\phi \circ \psi)^* = 0$. We have $\mathrm{Im}(\phi^*) \subset \mathrm{Ker}\left(\frac{df}{f}\Big|_Z\right)$ since

$$\left\langle \phi^*(w), \frac{df}{f}\Big|_Z \right\rangle = \langle \phi^*(w), \psi(\xi^*) \rangle = \langle \psi^*(\phi^*(w)), \xi^* \rangle = 0,$$

which implies $\mathrm{Im}(\phi^*) = \mathrm{Ker}\left(\frac{df}{f}\Big|_Z\right)$ by counting dimensions. In conclusion, we have the decomposition

$${}^bTM|_Z = \mathrm{Im}(\phi^*) \oplus \mathrm{Im}(i) = \mathrm{Ker}\left(\frac{df}{f}\Big|_Z\right) \oplus \mathbb{L}_Z \tag{5.18}$$

and isomorphisms

$$\phi^* : TZ \to \mathrm{Ker}\left(\frac{df}{f}\Big|_Z\right) \quad \text{and} \quad \rho|_Z : \mathrm{Ker}\left(\frac{df}{f}\Big|_Z\right) \to TZ \tag{5.19}$$

that are inverse to each other.

Now let $\omega \in \Gamma(\wedge^2({}^bT^*M))$ be a b-symplectic form on (M^{2n}, Z) and denote by $\Lambda \in \Gamma(\wedge^2({}^bTM))$ the inverse b-bivector field. The anchor $\rho : {}^bTM \to TM$ maps Λ to a log-symplectic structure $\Pi \in \Gamma(\wedge^2 TM)$. We denote by Π_Z the corank-one Poisson structure that is the restriction of Π to Z. Fix a defining function f for Z. By (5.18), we have

$$\wedge^2({}^bTM|_Z) = \wedge^2\mathrm{Ker}\left(\frac{df}{f}\Big|_Z\right) \oplus \left[\mathrm{Ker}\left(\frac{df}{f}\Big|_Z\right) \otimes \mathbb{L}_Z\right]$$

and therefore we can write

$$\begin{aligned}
\Lambda|_Z &= X \wedge g\xi + \Lambda_K \\
&= (gX) \wedge \xi + \Lambda_K \\
&= -\iota_{\frac{df}{f}\big|_Z} \Lambda|_Z \wedge \xi + \Lambda_K,
\end{aligned}$$

where ξ is the canonical non-vanishing section of $\mathbb{L}_Z$, $g \in C^\infty(Z), X \in \Gamma\left(\mathrm{Ker}\left(\frac{df}{f}\Big|_Z\right)\right)$ and $\Lambda_K \in \Gamma\left(\wedge^2\mathrm{Ker}\left(\frac{df}{f}\Big|_Z\right)\right)$. Moreover, we have:

i) Under the isomorphism $\wedge^2 TZ \cong \wedge^2\mathrm{Ker}\left(\frac{df}{f}\Big|_Z\right)$ arising from (5.19), we have that Λ_K corresponds with Π_Z. Indeed, if $\Lambda_K = \phi^*(w)$ for $w \in \Gamma(\wedge^2 TZ)$ then

$$\Pi_Z = \rho|_Z(\Lambda|_Z) = \rho|_Z(\Lambda_K) = \rho|_Z(\phi^*(w)) = w,$$

using that $\xi \in \Gamma(\mathrm{Ker}(\rho|_Z))$ in the second equality. Hence indeed $\phi^*(\Pi_Z) = \Lambda_K$.

[3]This is a general fact about short exact sequences in an additive category. We work in the additive category of vector bundles over Z, with as morphisms the bundle maps covering Id_Z.

ii) The b-vector field $-\iota_{\frac{df}{f}\big|_Z}\Lambda|_Z \in \Gamma\left(\operatorname{Ker}\left(\frac{df}{f}\Big|_Z\right)\right)$ corresponds under the isomorphism $\rho|_Z$ (5.19) to a vector field on Z, which we call v^f:

$$v^f := \rho|_Z\left(-\iota_{\frac{df}{f}\big|_Z}\Lambda|_Z\right). \tag{5.20}$$

Hence, for a fixed defining function f, we have that $\Lambda|_Z$ is completely determined by the bivector field Π_Z and the vector field v^f as

$$\Lambda|_Z = v^f \wedge \xi + \Pi_Z.$$

Remark 5.3.7. We make some remarks concerning the above observations.

i) Since df/f is a closed element of ${}^b\Omega^1(M)$, we have that $\rho\left(-\iota_{\frac{df}{f}}\Lambda\right)$ is a Poisson vector field (see the proof of Theorem 4.3.7). Its restriction to Z, which we denote by v^f, is then a Poisson vector field on Z since (Z, Π_Z) is a Poisson submanifold of (M, Π). Moreover, v^f is transverse to the symplectic leaves of (Z, Π_Z). Indeed, under the isomorphism $TZ \cong \operatorname{Ker}\left(\frac{df}{f}\Big|_Z\right)$, we write

$$\Lambda|_Z = v^f \wedge \xi + \Pi_Z,$$

and since Π is log-symplectic, we know that Π_Z has rank $2n-2$. Since $\Lambda \in \Gamma(\wedge^2({}^bTM))$ is non-degenerate, we have that

$$\Lambda|_Z^n = n v^f \wedge \xi \wedge \Pi_Z^{n-1}$$

is non-vanishing. In particular, $v^f \wedge \Pi_Z^{n-1}$ is non-vanishing, which implies that v^f is transverse to the leaves of (Z, Π_Z). This can be seen as follows. Choose splitting coordinates $(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, t)$ on Z, such that

$$\Pi_Z = \sum_{i=1}^{n-1} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}.$$

The expression of v^f in the coordinates $(x_1, y_1, \ldots, x_{n-1}, y_{n-1}, t)$ has to involve $\partial/\partial t$, since $v^f \wedge \Pi_Z^{n-1}$ is non-vanishing. As the leaves of Π_Z integrate the distribution

$$\operatorname{span}\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_{n-1}}, \frac{\partial}{\partial y_{n-1}}\right\},$$

it follows that v^f is transverse to the leaves of Π_Z.

ii) Changing defining function will change v^f by a Hamiltonian vector field. Indeed, any other defining function is of the form gf for some function g that is non-vanishing. Now note that

$$\frac{d(fg)}{fg}\bigg|_Z = \frac{df}{f}\bigg|_Z + \frac{dg}{g}\bigg|_Z = \frac{df}{f}\bigg|_Z + d\log(|g|)|_Z. \tag{5.21}$$

Recall that we consider usual de Rham forms as b-forms by pulling them back under the anchor map ρ. With this in mind, (5.21) gives that

$$-\iota_{\frac{d(fg)}{fg}\big|_Z}\Lambda|_Z = -\iota_{\frac{df}{f}\big|_Z}\Lambda|_Z - \iota_{\rho^*(d\log(|g|))|_Z}\Lambda|_Z,$$

whence

$$v^{fg} = v^f - \rho|_Z \left(\iota_{\rho^*(d\log(|g|))|_Z}\Lambda|_Z\right).$$

Now using that

$$\iota_{\rho^*(d\log(|g|))|_Z}\xi = 0 \quad \text{since } T_p^*Z = \langle \xi_p\rangle^0 \text{ at } p \in Z$$

along with the fact that $\rho|_Z(\xi) = 0$, we have

$$\begin{aligned}\rho|_Z \left(\iota_{\rho^*(d\log(|g|))|_Z}\Lambda|_Z\right) &= \rho|_Z \left(\iota_{\rho^*(d\log(|g|))|_Z}\Lambda|_K\right)\\ &= \iota_{d\log(|g|)|_Z}\rho(\Lambda_K)\\ &= \iota_{d\log(|g|)|_Z}(\Pi_Z)\\ &= \Pi_Z^\sharp\left(d\log(|g|)|_Z\right).\end{aligned}$$

Hence, we conclude

$$v^{fg} = v^f - X^{\mathrm{Ham}(\Pi_Z)}_{\log(|g|)|_Z}.$$

In the following, we denote by $[v] \in H^1_{\Pi_Z}(Z)$ the Poisson cohomology class $[v^f]$, for any choice of f.

iii) Replacing the b-symplectic form ω by $\varphi^*\omega$, where $\varphi : M \to M$ is a diffeomorphism such that $\varphi|_Z = \mathrm{Id}_Z$, the vector field v^f also changes by a Hamiltonian vector field. Indeed, noting that $\varphi^*(f)$ is also a defining function for Z, we have

$$\begin{aligned}v^{\varphi^*(f)}_{\varphi_*^{-1}\Lambda} &= \rho|_Z\left(-\iota_{\left.\frac{d(\varphi^*(f))}{\varphi^*(f)}\right|_Z}\left(\varphi_*^{-1}\Lambda\right)|_Z\right)\\ &= \rho|_Z\left(-\iota_{\left.\varphi^*\left(\frac{df}{f}\right)\right|_Z}\left(\varphi_*^{-1}\Lambda\right)|_Z\right)\\ &= \rho|_Z\left(\left(\varphi_*^{-1}|_Z\right)\left(-\iota_{\left.\frac{df}{f}\right|_Z}\Lambda|_Z\right)\right)\\ &= \rho|_Z\left(-\iota_{\left.\frac{df}{f}\right|_Z}\Lambda|_Z\right)\\ &= v^f_\Lambda,\end{aligned}$$

where the third equality holds by functoriality, and we used Lemma 5.3.5 in the fourth equality. By ii) we know that $v^{\varphi^*(f)}_{\varphi_*^{-1}\Lambda}$ and $v^f_{\varphi_*^{-1}\Lambda}$ differ by a Hamiltonian vector field, which then shows that v^f_Λ and $v^f_{\varphi_*^{-1}\Lambda}$ differ by a Hamiltonian vector field.

By the above remark, we obtain:

Proposition 5.3.8. *Let ω be a b-symplectic form on (M, Z), where M is orientable*[4] *and Π_Z is the induced corank-one Poisson structure on Z.*

i) Canonically associated to ω, there is a class in the Poisson cohomology $[v] \in H^1_{\Pi_Z}(Z)$, for which one (hence any) representative is transverse to the symplectic leaves of Z.

ii) If two b-symplectic structures inducing Π_Z are related by a diffeomorphism which is the identity on Z, then the associated cohomology classes agree. In other words, for each corank-one Poisson structure Π_Z on Z that arises from a b-symplectic form, the map

[4] Orientability of Z is automatic by Lemma 4.2.5

$$\{b-\textit{symplectic forms on } (M,Z) \textit{ inducing } \Pi_Z\}/\sim$$
$$\to \{\textit{elements of } H^1_{\Pi_Z}(Z) \textit{ transverse to the leaves}\}:$$
$$(\textit{class of } \omega) \mapsto [v] \tag{5.22}$$

is well-defined and canonical. Here $\sim$ is the equivalence relation by diffeomorphisms that are the identity on Z.

The assignment (5.22) can be described alternatively in terms of modular vector fields.

Proposition 5.3.9. *Let ω be a b-symplectic form on (M^{2n}, Z), where M is orientable. Let $\Lambda \in \Gamma(\wedge^2({}^bTM))$ be the b-bivector field inverse to ω, and let Π be the log-symplectic structure on M obtained by applying the anchor map $\rho : {}^bTM \to TM$ to Λ. Let $f : V \to \mathbb{R}$ be any defining function for Z, defined on a tubular neighborhood V of Z. Then there exists a volume form Ω on V such the vector field v^f is the modular vector field X^Ω_Π, restricted to Z.*

Proof. Since Π^n vanishes exactly on Z and vanishes linearly there, we have that $\chi := (1/f)\Pi^n$ is a nowhere vanishing $2n$-vector field on V. Setting Ω to be its dual $2n$-form (i.e. $\langle \chi, \Omega\rangle = 1$), we get that Ω is a volume form on V satisfying $\langle \Pi^n, \Omega\rangle = f$. The proof of Theorem 5.2.1 shows that we may use f as a global coordinate in the fibers of a tubular neighborhood U of Z, and we can decompose

$$\Lambda|_U = X^\Omega_\Pi|_Z \wedge f\frac{\partial}{\partial f} + \Pi_Z. \tag{5.23}$$

Using that $TZ = \mathrm{Ker}\left(\frac{df}{f}\Big|_Z\right)$ under the anchor map ρ, we therefore obtain

$$v^f = -\iota_{\frac{df}{f}\big|_Z}\left(X^\Omega_\Pi|_Z \wedge f\frac{\partial}{\partial f} + \Pi_Z\right) = X^\Omega_\Pi|_Z.$$

∎

We showed in Proposition 2.9.6 that if Ω is a volume form and h a non-vanishing function, then

$$X^{h\Omega}_\Pi = X^\Omega_\Pi - X_{\log|h|}.$$

Therefore the assignment (5.22) in Proposition 5.3.8 can be described as follows: to the b-symplectic form ω we associate $[X^\Omega_\Pi\big|_Z] \in H^1_{\Pi_Z}(Z)$, where Π is the log-symplectic structure corresponding with ω, and Ω is any volume form on M.

Bijectivity of the correspondence

Restricting to a tubular neighborhood of Z, the assigment (5.22) becomes a bijection. Injectivity is shown by the theorem below, which appeared in [GMP2, Theorem 35]. However, the proof given there is sloppy regarding the volume forms used, and moreover it contains a gap in a crucial place. See Remark 5.3.12 below. We present a more elaborate argument that rectifies these problems.

Theorem 5.3.10. *Let ω_0 and ω_1 be b-symplectic forms on (M^{2n}, Z), where M is orientable. Let $\Pi_0, \Pi_1 \in \Gamma(\wedge^2 TM)$ be the corresponding log-symplectic structures. Assume that we have $\Pi_0|_Z = \Pi_1|_Z := \Pi_Z \in \wedge^2(TZ)$, and suppose moreover that*

$$X^\Omega_{\Pi_1}|_Z = X^\Omega_{\Pi_0}|_Z + X_f^{Ham(\Pi_Z)},$$

where Ω is some volume form on M and $X_f^{Ham(\Pi_Z)} \in \mathfrak{X}(Z)$ is the Hamiltonian vector field for Π_Z associated with the function $f \in C^\infty(Z)$. Then there exist neighborhoods O_0, O_1 of Z in M and a diffeomorphism $\gamma : O_0 \to O_1$ such that $\gamma|_Z = Id_Z$ and $\gamma^*\omega_1 = \omega_0$.

Proof.

Step 1

Let g be an extension of the function f, defined in some neighborhood E of Z. Consider the volume form $\Omega' := e^{-g}\Omega$ on E. We then have on E:

$$X_{\Pi_0}^{\Omega'} = X_{\Pi_0}^{\Omega} - X_{\log(e^{-g})}^{\text{Ham}(\Pi_0)} = X_{\Pi_0}^{\Omega} - X_{-g}^{\text{Ham}(\Pi_0)} = X_{\Pi_0}^{\Omega} + X_g^{\text{Ham}(\Pi_0)}.$$

Using that (Z, Π_Z) is a Poisson submanifold of (M, Π_0), we have

$$X_g^{\text{Ham}(\Pi_0)}|_Z = \left(\Pi_0^\sharp(dg)\right)\Big|_Z = \Pi_Z^\sharp(d(g|_Z)) = \Pi_Z^\sharp(df) = X_f^{\text{Ham}(\Pi_Z)}.$$

Hence

$$X_{\Pi_1}^{\Omega}|_Z = X_{\Pi_0}^{\Omega}|_Z + X_f^{\text{Ham}(\Pi_Z)} = X_{\Pi_0}^{\Omega'}|_Z. \tag{5.24}$$

Step 2

Using the Moser trick for volume forms, we will now find a diffeomorphism $\psi : V_0 \to V_1$, where V_0, V_1 are open neighborhoods of Z, such that $\psi^*\Omega' = \Omega$ and $\psi|_Z = \text{Id}_Z$.

Put $\Omega_0 = \Omega$ and $\Omega_1 = \Omega'$. Consider the straight line homotopy

$$\Omega_t := \Omega_0 + t(\Omega_1 - \Omega_0) \qquad 0 \le t \le 1.$$

We claim that Ω_t is a volume form for each $t \in [0,1]$. Note that

$$\Omega_t = \Omega + t\left(e^{-g}\Omega - \Omega\right) = \left(1 + t(e^{-g} - 1)\right)\Omega,$$

where $1 + t(e^{-g} - 1)$ is nowhere vanishing for all $t \in [0,1]$. Indeed, for $t = 0$ it is clearly non-vanishing, whereas for $t \neq 0$ we have

$$1 + t(e^{-g(p)} - 1) = 0 \Leftrightarrow e^{-g(p)} = \frac{-1}{t} + 1 \le 0.$$

Here the last inequality holds since $0 < t \le 1$. So we would have that $e^{-g(p)} \le 0$, which is impossible. We now have a path of volume forms Ω_t on E. Note that $\Omega_1 - \Omega_0$ is closed, being a differential form of top degree. Moreover, its pullback to Z vanishes, being a $2n$-form on a $2n-1$-dimensional manifold. Hence the Relative Poincaré Lemma applies (see Proposition 1.3.9), which tells us that there exists $\nu \in \Omega^{2n-1}(E)$ such that

$$\Omega_1 - \Omega_0 = d\nu$$

and $\nu|_Z = 0$. To find the desired diffeomorphism ψ, it now suffices to solve the Moser equation

$$\iota_{X_t}\Omega_t = -\nu \qquad (0 \le t \le 1)$$

for X_t, which is possible by Lemma 5.3.11 below. Note that $X_t|_Z = 0$ for each $t \in [0,1]$, because $\nu|_Z = 0$. We now integrate $\{X_t\}_{t\in[0,1]}$ to an isotopy $\{\rho_t\}_{t\in[0,1]}$, and application of the Tube Lemma as in the Local Moser Theorem 1.3.14 ensures the existence of an open neighborhood $V \subset E$ of Z such that

$$\rho : [0,1] \times V \to E,$$

i.e. ρ_t is defined on V for each $t \in [0,1]$. Note that $\rho_t|_Z = \text{Id}_Z$ since $X_t|_Z = 0$. Putting $\psi = \rho_1$, $V_0 = V$ and $V_1 = \rho_1(V)$, we obtain the desired diffeomorphism $\psi : V_0 \to V_1$ satisfying

$$\psi^*\Omega' = \Omega \qquad \text{and} \qquad \psi|_Z = \text{Id}_Z.$$

Step 3

Using that $\psi^*\Omega' = \Omega$, as obtained in the previous step, we have

$$\begin{aligned} X^{\Omega}_{-(\psi^*\omega_0)^{-1}}\Big|_Z &= X^{\psi^*\Omega'}_{-(\psi^*\omega_0)^{-1}}\Big|_Z = \left(\psi_*^{-1}\left(X^{\Omega'}_{-\omega_0^{-1}}\right)\right)\Big|_Z = \left(\psi_*^{-1}\left(X^{\Omega'}_{\Pi_0}\right)\right)\Big|_Z \\ &= \left(\psi^{-1}|_Z\right)_*\left(X^{\Omega'}_{\Pi_0}\Big|_Z\right) = X^{\Omega'}_{\Pi_0}\Big|_Z = X^{\Omega}_{\Pi_1}|_Z\,. \end{aligned} \tag{5.25}$$

The second equality holds by functoriality, the last equality is (5.24) and we used in addition that $X^{\Omega'}_{\Pi_0}$ is tangent to Z at point of Z. Now consider the log-symplectic structures $\widetilde{\Pi_0} := -(\psi^*\omega_0)^{-1}$ and Π_1, defined on a neighborhood of Z. Since $\psi_*\left(\widetilde{\Pi_0}\right) = \Pi_0$, we have

$$\Pi_1|_Z = \Pi_0|_Z = \left(\psi_*\left(\widetilde{\Pi_0}\right)\right)\Big|_Z = (\psi|_Z)_*\left(\widetilde{\Pi_0}\Big|_Z\right) = \widetilde{\Pi_0}\Big|_Z \in \wedge^2(TZ),$$

and the modular vector fields of $\widetilde{\Pi_0}$ and Π_1 with respect to Ω coincide on Z.

Step 4

By Theorem 5.2.1, there exists a tubular neighborhood $U \subset Z \times \mathbb{R}$ of Z on which

$$\begin{aligned} \widetilde{\Pi_0}\Big|_U &= X^{\Omega}_{\widetilde{\Pi_0}}\Big|_Z \wedge t_0\frac{\partial}{\partial t_0} + \widetilde{\Pi_0}|_Z \\ \Pi_1|_U &= X^{\Omega}_{\Pi_1}|_Z \wedge t_1\frac{\partial}{\partial t_1} + \Pi_1|_Z, \end{aligned} \tag{5.26}$$

where $t_0 = \langle \widetilde{\Pi_0}^n, \Omega\rangle$ and $t_1 = \langle \Pi_1^n, \Omega\rangle$ are defining functions for Z. Considering $\widetilde{\Pi_0}^n$ and Π_1^n as nowhere vanishing sections of the line bundle $\wedge^{2n}({}^bTM)$, there exists a nowhere vanishing function $f \in C^\infty(U)$ with $\widetilde{\Pi_0}^n = f\Pi_1^n$. This implies that $t_0 = ft_1$. Using t_0 as defining function for Z, we can decompose the b-symplectic forms $\widetilde{\omega_0}$ and ω_1 dual to $\widetilde{\Pi_0}$ and Π_1 as

$$\begin{aligned} \widetilde{\omega_0}|_U &= \alpha_0 + \frac{dt_0}{t_0} \wedge p^*(\theta_0) \\ \omega_1|_U &= \alpha_1 + \frac{dt_0}{t_0} \wedge p^*(\theta_1), \end{aligned}$$

where $p : U \to Z$ is the projection.

<u>Claim:</u> $\widetilde{\omega_0}|_Z = \omega_1|_Z$.

By Lemma 5.1.2, we know that $(\theta_0, \widetilde{\alpha_0})$ is the cosymplectic structure corresponding to $\left(\widetilde{\Pi_0}\Big|_Z, \widetilde{\Pi_0}^\sharp\left(\frac{dt_0}{t_0}\right)\Big|_Z\right)$, and $(\theta_1, \widetilde{\alpha_1})$ corresponds to $\left(\Pi_1|_Z, \Pi_1^\sharp\left(\frac{dt_0}{t_0}\right)\Big|_Z\right)$. Since

$$\frac{dt_0}{t_0} = \frac{df}{f} + \frac{dt_1}{t_1},$$

we have, using (5.26) and the fact that the modular vector fields $X^{\Omega}_{\widetilde{\Pi_0}}\Big|_Z$ and $X^{\Omega}_{\Pi_1}|_Z$ as well as $\widetilde{\Pi_0}|_Z$ and $\Pi_1|_Z$ are tangent to Z:

$$\Pi_1^\sharp\left(\frac{dt_0}{t_0}\right)\Big|_Z = -\,X^{\Omega}_{\Pi_1}|_Z\,\frac{dt_0}{t_0}\left(t_1\frac{\partial}{\partial t_1}\right)\Big|_Z = -\,X^{\Omega}_{\Pi_1}|_Z\left(\frac{df}{f}\Big|_Z\left(t_1\frac{\partial}{\partial t_1}\Big|_Z\right) + 1\right).$$

Note here that df/f is an honest de Rham form since f is non-vanishing. Hence at points of Z, it annihilates the normal b-vector field $t_1\partial/\partial t_1$. So

$$\Pi_1^\sharp\left(\frac{dt_0}{t_0}\right)\bigg|_Z = -\left.X^\Omega_{\Pi_1}\right|_Z = -\left.X^\Omega_{\widetilde{\Pi_0}}\right|_Z = \widetilde{\Pi_0}^\sharp\left(\frac{dt_0}{t_0}\right)\bigg|_Z.$$

Hence $\left(\left.\widetilde{\Pi_0}\right|_Z, \widetilde{\Pi_0}^\sharp\left(\frac{dt_0}{t_0}\right)\Big|_Z\right) = \left(\left.\Pi_1\right|_Z, \Pi_1^\sharp\left(\frac{dt_0}{t_0}\right)\Big|_Z\right)$, which implies that $(\theta_0, \widetilde{\alpha_0}) = (\theta_1, \widetilde{\alpha_1})$. We conclude that

$$\left.\widetilde{\omega_0}\right|_Z = \widetilde{\alpha_0} + \left.\frac{dt_0}{t_0}\right|_Z \wedge \theta_0 = \widetilde{\alpha_1} + \left.\frac{dt_0}{t_0}\right|_Z \wedge \theta_1 = \left.\omega_1\right|_Z.$$

Step 5

The Local b-Moser Theorem 4.2.7 gives open neighborhoods O_0 and O_1 of Z and a diffeomorphism $\phi : O_0 \to O_1$ such that $\phi|_Z = \mathrm{Id}_Z$ and $\phi^*\widetilde{\omega_0} = \omega_1$. Since $\widetilde{\omega_0} = \psi^*\omega_0$, it follows that

$$\phi^*(\psi^*\omega_0) = (\psi\circ\phi)^*\omega_0 = \omega_1,$$

where $\psi\circ\phi : O_0 \cap \phi^{-1}(V_0) \to \psi(O_1)\cap V_1$ is a diffeomorphism between open neighborhoods of Z, satisfying $(\psi\circ\phi)|_Z = \mathrm{Id}_Z$. This finishes the proof.

□

Lemma 5.3.11. *If V is a real vectorspace of dimension n and $\mu \in \wedge^n V^*$ is nonzero, then the linear map*

$$V \to \wedge^{n-1}V^* : v \mapsto \iota_v\mu$$

is an isomorphism.

Proof. It is enough to show injectivity, since

$$\dim\left(\wedge^{n-1}V^*\right) = \binom{n}{n-1} = n = \dim(V).$$

Suppose $\iota_v\mu = 0$ and assume by contradiction that $v \neq 0$. We extend $\{v\}$ to a basis $\{v, v_2, \dots, v_n\}$ of V and obtain that

$$\mu(v, v_2, \dots, v_n) = (\iota_v\mu)(v_2, \dots, v_n) = 0.$$

So μ evaluates a basis of V to zero, which implies that $\mu = 0$. This contradicts the assumption of the lemma. □

Remark 5.3.12. In [GMP2, Theorem 35], one claims that the assumptions of Theorem 5.3.10 imply that $\omega_0|_Z = \omega_1|_Z$. This is not true: we give a concrete counterexample below, which is a slight adaptation of [GMP2, Example 18]. So the proof of Theorem 5.3.10 is more subtle than suggested in [GMP2, Theorem 35]. The crucial ingredient missing there is the Moser argument for volume forms that we use in Step 2 of our proof. It results in a diffeomorphism ψ defined near Z so that

$$(\psi^*\omega_0)|_Z = \omega_1|_Z,$$

and at this stage we can safely use the local b-Moser theorem. This was not possible from the outset, since $\omega_0|_Z \neq \omega_1|_Z$ in general.
As a counterexample to the claim in [GMP2, Theorem 35], we consider $S^2 \times (S^1 \times \mathbb{R})$, with cylindrical coordinates (θ, h) on S^2 and (θ_1, z) on $S^1\times\mathbb{R}$. Consider the log-symplectic structures

$$\Pi = h\frac{\partial}{\partial h}\wedge\left(\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\theta_1}\right) + \frac{\partial}{\partial\theta_1}\wedge\frac{\partial}{\partial z},$$

$$\widetilde{\Pi} = h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial z}.$$

On their common singular locus $Z \leftrightarrow \{h = 0\}$, they both induce the Poisson structure

$$\Pi_Z := \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial z}.$$

The b-symplectic forms $\omega := -\Pi^{-1}$ and $\widetilde{\omega} := -\left(\widetilde{\Pi}\right)^{-1}$ are given by

$$\begin{aligned}
\omega &= \frac{dh}{h} \wedge d\theta + dz \wedge d\theta + d\theta_1 \wedge dz, \\
\widetilde{\omega} &= \frac{dh}{h} \wedge d\theta + d\theta_1 \wedge dz.
\end{aligned}$$

We take the volume form $\Omega := d\theta_1 \wedge dz \wedge dh \wedge d\theta$. By Lemma 4.2.14, we then know that

$$X^{\Omega}_{\widetilde{\Pi}} = -\frac{\partial}{\partial \theta}.$$

We now compute X^{Ω}_{Π}. For any smooth function f, we have

$$X_f = h\frac{\partial f}{\partial h}\frac{\partial}{\partial \theta} - h\frac{\partial f}{\partial \theta}\frac{\partial}{\partial h} + h\frac{\partial f}{\partial h}\frac{\partial}{\partial \theta_1} - h\frac{\partial f}{\partial \theta_1}\frac{\partial}{\partial h} + \frac{\partial f}{\partial \theta_1}\frac{\partial}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial}{\partial \theta_1},$$

using Lemma 2.7.4. We then note that

$$\begin{aligned}
\pounds_{X_f} d\theta_1 &= d\left(\pounds_{X_f}\theta_1\right) = d\left(d\theta_1(X_f)\right) = d\left(h\frac{\partial f}{\partial h} - \frac{\partial f}{\partial z}\right), \\
\pounds_{X_f} dz &= d\left(\pounds_{X_f} z\right) = d\left(dz(X_f)\right) = d\left(\frac{\partial f}{\partial \theta_1}\right), \\
\pounds_{X_f} dh &= d\left(\pounds_{X_f} h\right) = d\left(dh(X_f)\right) = d\left(-h\frac{\partial f}{\partial \theta} - h\frac{\partial f}{\partial \theta_1}\right), \\
\pounds_{X_f} d\theta &= d\left(\pounds_{X_f}\theta\right) = d\left(d\theta(X_f)\right) = d\left(h\frac{\partial f}{\partial h}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\pounds_{X_f}\Omega &= d\left(h\frac{\partial f}{\partial h} - \frac{\partial f}{\partial z}\right) \wedge dz \wedge dh \wedge d\theta + d\theta_1 \wedge d\left(\frac{\partial f}{\partial \theta_1}\right) \wedge dh \wedge d\theta \\
&\quad + d\theta_1 \wedge dz \wedge d\left(-h\frac{\partial f}{\partial \theta} - h\frac{\partial f}{\partial \theta_1}\right) \wedge d\theta + d\theta_1 \wedge dz \wedge dh \wedge d\left(h\frac{\partial f}{\partial h}\right) \\
&= \left(h\frac{\partial^2 f}{\partial \theta_1 \partial h} - \frac{\partial^2 f}{\partial \theta_1 \partial z} + \frac{\partial^2 f}{\partial z \partial \theta_1} - \frac{\partial f}{\partial \theta} - h\frac{\partial^2 f}{\partial h \partial \theta} - \frac{\partial f}{\partial \theta_1} - h\frac{\partial^2 f}{\partial h \partial \theta_1} + h\frac{\partial^2 f}{\partial \theta \partial h}\right)\Omega \\
&= \left(-\frac{\partial f}{\partial \theta} - \frac{\partial f}{\partial \theta_1}\right)\Omega,
\end{aligned}$$

so that

$$X^{\Omega}_{\Pi} = -\frac{\partial}{\partial \theta} - \frac{\partial}{\partial \theta_1}.$$

We note that

$$X^{\Omega}_{\Pi}\big|_Z - X^{\Omega}_{\widetilde{\Pi}}\Big|_Z = -\frac{\partial}{\partial \theta_1} = \Pi_Z^{\sharp}(dz)$$

is a Hamiltonian vector field for Π_Z, but nonetheless

$$\widetilde{\omega}|_Z \neq \omega|_Z.$$

We can now prove our main statement, which refines [GMP2, Theorem 50].

Theorem 5.3.13. *Let (M, Z) be a b-manifold with M orientable, and assume that Π_Z is a corank-one Poisson structure on Z that is induced by a b-symplectic structure. Then the assignment*

$$\begin{aligned}\{b-\textit{symplectic forms defined on a tubular neighborhood of } Z \textit{ inducing } \Pi_Z\}/\sim \\ \to \{\textit{elements of } H^1_{\Pi_Z}(Z) \textit{ transverse to the leaves}\}: \\ (\textit{class of } \omega) \mapsto [v] \end{aligned} \tag{5.27}$$

is bijective. Here $\sim$ is the equivalence relation by diffeomorphisms defined in a neighborhood of Z that are the identity on Z.

Proof. Injectivity of (5.27) is proved in Theorem 5.3.10. Surjectivity is readily checked as follows. By orientability of Z and M, the normal bundle NZ is trivial, so we can choose a tubular neighborhood $U \cong Z \times (-1, 1)$, with coordinate t on the interval. Now if v is a transverse Poisson vector field on (Z, Π_Z), then

$$\Pi := v \wedge t\frac{\partial}{\partial t} + \Pi_Z \tag{5.28}$$

is a log-symplectic structure on U with singular locus Z, inducing Π_Z on Z. Moreover, the vector field v^t_Π as defined in (5.20) is equal to v, which proves surjectivity. □

We finish this chapter by working out an example on Theorems 5.3.1 and 5.3.13.

Example 5.3.14. Consider the b-manifold $(M, Z) = (\mathbb{R}^2, \{y = 0\})$. We start with the zero Poisson structure $\Pi_Z \equiv 0$ on Z (which is the only Poisson structure on Z).

- Does $\Pi_Z = 0$ come from a log-symplectic structure? By Theorem 5.3.1, we have to check if there exists a Poisson vector field on Z transverse to the leaves of Z. Since any vector field on Z is Poisson and the leaves of Z are its points, a transverse Poisson vector field is the same thing as a nowhere vanishing vector field. For sure, such a vector field exists on Z and therefore $\Pi_Z = 0$ is induced by a log-symplectic structure.
- The elements of $H^1_{\Pi_Z}(Z) = \mathfrak{X}(Z)$ that are transverse to the symplectic leaves of Z are given by
$$\left\{ g(x)\frac{\partial}{\partial x} : \ g \in C^\infty(Z) \text{ non-vanishing} \right\}.$$
To each of these vector fields corresponds a class of b-symplectic structures inducing Π_Z, by Theorem 5.3.13. Note that (5.28) shows us how to find the class of b-symplectic structures corresponding with $g \in C^\infty(Z)$, namely we take the equivalence class of
$$\omega_g := -\left(g(x)\frac{\partial}{\partial x} \wedge y\frac{\partial}{\partial y} \right)^{-1} = g^{-1}(x)dx \wedge \frac{dy}{y} \in {}^b\Omega^2(M).$$
Let us double-check that, if g and g' are different non-vanishing functions on Z, then the corresponding b-symplectic forms ω_g and $\omega_{g'}$ are indeed not related by a diffeomorphism $\rho : M \to M$ with $\rho|_Z = \mathrm{Id}_Z$. If ρ is such a diffeomorphism, then Lemma 5.3.5 implies that
$${}^b\rho_*|_Z \left(\left(y\frac{\partial}{\partial y} \right)\Big|_Z \right) = \left(y\frac{\partial}{\partial y} \right)\Big|_Z$$

and

$$ {}^b\rho_*|_Z \left(\left.\frac{\partial}{\partial x}\right|_Z \right) = \left.\left(\frac{\partial}{\partial x} + h(x) y \frac{\partial}{\partial y} \right)\right|_Z , $$

for some function $h \in C^\infty(Z)$. We then have for $\omega \in {}^b\Omega^2(M)$:

$$ \begin{aligned} (\rho^*\omega)|_Z \left(\left.\frac{\partial}{\partial x}\right|_Z , \left.\left(y\frac{\partial}{\partial y}\right)\right|_Z \right) &= \omega|_Z \left({}^b\rho_*|_Z \left(\left.\frac{\partial}{\partial x}\right|_Z \right), {}^b\rho_*|_Z \left(\left.\left(y\frac{\partial}{\partial y}\right)\right|_Z \right) \right) \\ &= \omega|_Z \left(\left.\left(\frac{\partial}{\partial x} + h(x) y \frac{\partial}{\partial y} \right)\right|_Z , \left.\left(y\frac{\partial}{\partial y}\right)\right|_Z \right) \\ &= \omega|_Z \left(\left.\frac{\partial}{\partial x}\right|_Z , \left.\left(y\frac{\partial}{\partial y}\right)\right|_Z \right), \end{aligned} $$

where the last equality holds by skew-symmetry of ω. Hence $(\rho^*\omega)|_Z = \omega|_Z$. This shows that for $g \neq g'$, the b-symplectic forms ω_g and $\omega_{g'}$ are not related by such a diffeomorphism ρ: if this were the case, then ω_g and $\omega_{g'}$ would have the same restriction to Z, which they don't.

- So we conclude that there are as many pairwise inequivalent log-symplectic extensions of $\Pi_Z = 0$ as there are non-vanishing functions on the real line. So the set of equivalence classes of log-symplectic extensions of Π_Z corresponds to an open subset of the infinite dimensional real vector space $C^\infty(\mathbb{R})$.

Chapter 6

Foliation invariants

This chapter aims to give a characterization of a certain class of compact corank-one Poisson manifolds, namely those equipped with a closed one-form defining the symplectic foliation and a closed two-form extending the symplectic form on each leaf.

To do so, we will define two foliation invariants, the vanishing of which is equivalent with the existence of such a closed defining one- and two-form. We then show that a foliation with vanishing invariants on a compact manifold M is defined by a fibration over S^1, and we will characterize the manifold M as a mapping torus.

The symplectic foliation on the singular locus of a log-symplectic structure always has vanishing invariants, so that the aforementioned results apply in particular to the singular loci of log-symplectic structures, provided they are compact. This chapter follows [GMP1].

6.1 Introducing two foliation invariants

6.1.1 The first obstruction class

Throughout, we will be dealing with regular codimension-one foliations. Let us briefly recall what a regular foliation is.

Definition 6.1.1. A regular foliation $\mathcal{F}$ of dimension k on a manifold M^n is a decomposition of M into connected immersed submanifolds $\{L_a\}_{a\in A}$ of dimension k, called the leaves of the foliation, with the following local property: every point in M has a neighborhood U with coordinates $(x_1, \ldots, x_n)$ such for each leaf L_a, the connected components of $U \cap L_a$ are given by the equations

$$\begin{cases} x_{k+1} = \text{constant} \\ \quad\vdots \\ x_n = \text{constant} \end{cases}.$$

Such charts $(U, x_1, \ldots, x_n)$ are called foliated charts. The codimension of $\mathcal{F}$ is $n-k$.

So a regularly foliated manifold M is locally modelled as an affine space decomposed into parallel affine subspaces.

Definition 6.1.2. Let $\mathcal{F}$ be a regular foliation on M. The union of the tangent spaces T_pL for $p \in M$, where L is the leaf through p, forms a subbundle $T\mathcal{F} \subset TM$. The normal bundle of the foliation is the quotient bundle $TM/T\mathcal{F}$. The conormal bundle is its dual bundle $(TM/T\mathcal{F})^*$, which is identified with the annihilator

$$\mathrm{Ann}(T\mathcal{F}) = \{\alpha \in T_p^*M : p \in M,\ \alpha(v) = 0\ \ \forall v \in T_pL\}.$$

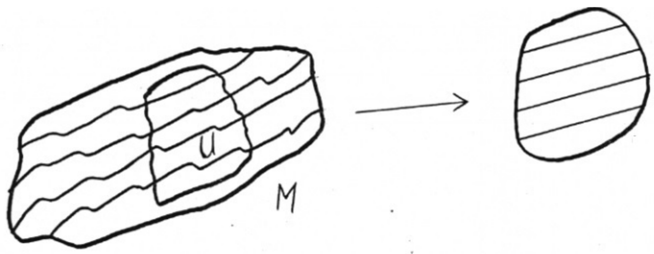


Figure 6.1: A regular 1*-dimensional foliation. Figure taken from [Mil]*

Definition 6.1.3. A foliation $\mathcal{F}$ on M is called transversely orientable[1] if the normal bundle $TM/T\mathcal{F}$ is orientable.

Intuitively, a transversely orientable foliation $\mathcal{F}$ is a foliation that allows one to distinguish between "above the leaf" and "below the leaf". In case $\mathcal{F}$ is a codimension-one foliation, we have the following equivalences:

$$\begin{aligned}\mathcal{F} \text{ is transversely orientable} &\Leftrightarrow TM/T\mathcal{F} \text{ is orientable}\\ &\Leftrightarrow (TM/T\mathcal{F})^* \text{ has a nowhere vanishing section}\end{aligned}$$

A nowhere vanishing section of $(TM/T\mathcal{F})^*$ is what we call a defining one-form for the codimension-one foliation $\mathcal{F}$.

Definition 6.1.4. Let $\mathcal{F}$ be a transversely orientable codimension-one foliation of M. A differential form $\alpha \in \Omega^1(M)$ is a defining one-form of the foliation $\mathcal{F}$ if it is nowhere vanishing and $i_L^*\alpha = 0$ for all leaves L, where $i_L : L \hookrightarrow M$ is the inclusion.

Being non-vanishing sections of the line bundle $(TM/T\mathcal{F})^*$, any two defining one-forms differ by a non-vanishing factor in $C^\infty(M)$.

Remark 6.1.5. If the foliation $\mathcal{F}$ is induced by an orientable log-symplectic structure Π on some ambient manifold, then we can choose a defining-one form α such that $\alpha(X_\Pi^\Omega\big|_M) = 1$ by Theorem 5.1.1. With this extra condition, the defining one-form is unique, even when we consider a different volume form Ω: this causes the modular vector field to change by a Hamiltonian vector field, which is tangent to the leaves of $\mathcal{F}$.

A basic property of defining one-forms is the following.

Lemma 6.1.6. *Let $\mathcal{F}$ be a codimension-one foliation of M^n, with defining one-form $\alpha \in \Omega^1(M)$. Then for $\mu \in \Omega^k(M)$, we have $\mu \in \alpha \wedge \Omega^{k-1}(M) \Leftrightarrow \alpha \wedge \mu = 0$.*

Proof. One direction is clear, for if $\mu = \alpha \wedge \eta$ for some $\eta \in \Omega^{k-1}(M)$ then

$$\alpha \wedge \mu = \alpha \wedge \alpha \wedge \eta = 0.$$

Conversely, assume that $\alpha \wedge \mu = 0$. Fix a point $p \in M$ and choose a foliated chart $(U, x_1, \ldots, x_n)$ around p so that the connected components of the leaves are locally given by x_n = constant. We write

$$\alpha|_U = \sum_{i=1}^n g_i dx_i,$$

[1]The terminology "co-orientable" is also used.

and since the pullback of α to each leaf is zero, we have

$$g_j = \alpha|_U\left(\frac{\partial}{\partial x_j}\right) = 0 \text{ for } j = 1, \ldots, n-1.$$

Hence $\alpha|_U = g_n dx_n$, with g_n non-vanishing. Now write

$$\mu|_U = \sum_{1 \le i_1 < \cdots < i_k \le n} f_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Since $\alpha \wedge \mu = 0$, we have that $g_n f_{i_1,\ldots,i_k} = 0$ whenever $n \notin \{i_1, \ldots, i_k\}$. As g_n is non-vanishing, this implies that $f_{i_1,\ldots,i_k} = 0$ whenever $n \notin \{i_1, \ldots, i_k\}$. Therefore,

$$\begin{aligned}
\mu|_U &= \sum_{1 \le i_1 < \cdots < i_{k-1} < n} f_{i_1,\ldots,i_{k-1},n} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \wedge dx_n \\
&= \left(\sum_{1 \le i_1 < \cdots < i_{k-1} < n} f_{i_1,\ldots,i_{k-1},n} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \right) \wedge dx_n \\
&= \left(\sum_{1 \le i_1 < \cdots < i_{k-1} < n} \frac{f_{i_1,\ldots,i_{k-1},n}}{g_n} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \right) \wedge \alpha|_U \\
&= \alpha|_U \wedge \left((-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_{k-1} < n} \frac{f_{i_1,\ldots,i_{k-1},n}}{g_n} dx_{i_1} \wedge \cdots \wedge dx_{i_{k-1}} \right) \\
&:= \alpha|_U \wedge \eta_U. \qquad (6.1)
\end{aligned}$$

We make a covering $\mathcal{U}$ of M consisting of such opens U, and we choose a partition of unity $\{\phi_U : U \in \mathcal{U}\}$ subordinate to $\mathcal{U}$. If we let

$$\eta := \sum_{U \in \mathcal{U}} \phi_U \eta_U \in \Omega^{k-1}(M),$$

then (6.1) implies that $\mu = \alpha \wedge \eta$. □

Corollary 6.1.7. *Let $\mathcal{F}$ be a codimension-one foliation of M^n with defining one-form α. Then for $\mu \in \Omega^k(M)$, we have that $\mu \in \alpha \wedge \Omega^{k-1}(M)$ if and only if $i_L^* \mu = 0$ for each leaf $L \in \mathcal{F}$.*

Proof. If $\mu = \alpha \wedge \eta$ for some $\eta \in \Omega^{k-1}(M)$, then

$$i_L^* \mu = (i_L^* \alpha) \wedge (i_L^* \eta) = 0 \wedge (i_L^* \eta) = 0.$$

Conversely, assume that $i_L^* \mu = 0$ for each leaf $L \in \mathcal{F}$. By Lemma 6.1.6, it suffices to show that $\alpha \wedge \mu = 0$. Choose $p \in M$ and let L be the leaf through p. We have to check that $(\alpha \wedge \mu)_p$ vanishes on $(k+1)$-tuples $(v_1, \ldots, v_{k+1})$ where either $v_j \in T_pL$ for all $j \in \{1, \ldots, k+1\}$, or $v_1, \ldots, v_k \in T_pL$ and $v_{k+1} \notin T_pL$. In the former case, we have

$$\begin{aligned}
(\alpha \wedge \mu)_p(v_1, \ldots, v_{k+1}) &= \left(i_L^*(\alpha \wedge \mu)\right)_p (v_1, \ldots, v_{k+1}) \\
&= \left((i_L^* \alpha) \wedge (i_L^* \mu)\right)_p (v_1, \ldots, v_{k+1}) \\
&= 0.
\end{aligned}$$

In the latter case,

$$(\alpha \wedge \mu)_p(v_1, \ldots, v_{k+1}) = \frac{1}{k!} \sum_{\sigma \in S_{k+1}} \operatorname{sgn}(\sigma) \alpha_p(v_{\sigma(1)}) \mu_p(v_{\sigma(2)}, \ldots, v_{\sigma(k+1)}). \qquad (6.2)$$

If $\sigma(1) \neq k+1$ then $\alpha_p(v_{\sigma(1)}) = 0$ since $i_L^*\alpha = 0$. If $\sigma(1) = k+1$, then $\mu_p(v_{\sigma(2)}, \ldots, v_{\sigma(k+1)}) = 0$ since $i_L^*\mu = 0$. Hence the expression (6.2) vanishes, and we conclude that $(\alpha \wedge \mu)_p = 0$. □

Remark 6.1.8. In particular, if $\alpha \in \Omega^1(M)$ is a defining one-form for $\mathcal{F}$, then

$$i_L^*(d\alpha) = d(i_L^*\alpha) = 0,$$

so that

$$d\alpha = \beta \wedge \alpha \tag{6.3}$$

for some $\beta \in \Omega^1(M)$. This implies that $\alpha \wedge \Omega^{\bullet-1}(M)$ is a subcomplex of $\Omega^\bullet(M)$. Indeed, for $\alpha \wedge \eta \in \alpha \wedge \Omega^{k-1}(M)$ one has

$$d(\alpha \wedge \eta) = (d\alpha) \wedge \eta - \alpha \wedge (d\eta) = -\alpha \wedge \beta \wedge \eta - \alpha \wedge (d\eta) = \alpha \wedge (-\beta \wedge \eta - d\eta) \in \alpha \wedge \Omega^k(M).$$

In fact, the complex $\alpha \wedge \Omega^{\bullet-1}(M)$ does not depend on the choice of defining one-form α. Indeed, if α' is another defining one-form, then $\alpha' = f\alpha$ for some non-vanishing function $f \in C^\infty(M)$ and one can write

$$\alpha \wedge \eta = \alpha' \wedge \frac{1}{f}\eta,$$
$$\alpha' \wedge \xi = \alpha \wedge (f\xi),$$

showing that $\alpha \wedge \Omega^{\bullet-1}(M) = \alpha' \wedge \Omega^{\bullet-1}(M)$[2]. Therefore, with a transversely orientable codimension-one foliation $\mathcal{F}$ on M comes canonically a short exact sequence of complexes

$$0 \longrightarrow \big(\alpha \wedge \Omega^{\bullet-1}(M), d\big) \longrightarrow (\Omega^\bullet(M), d) \xrightarrow{j} \left(\frac{\Omega^\bullet(M)}{\alpha \wedge \Omega^{\bullet-1}(M)}, \overline{d}\right) \longrightarrow 0. \tag{6.4}$$

The quotient complex $\big(\Omega^\bullet(M)/\alpha \wedge \Omega^{\bullet-1}(M), \overline{d}\big)$ is nothing else but the complex of differential forms along the leaves of $\mathcal{F}$. Indeed, let D denote the tangent distribution of $\mathcal{F}$. We then have a surjective map

$$\Omega^\bullet(M) \to \Gamma(\wedge^\bullet D^*) : \eta \mapsto \eta|_D,$$

whose kernel is

$$\{\eta \in \Omega^\bullet(M) : i_L^*\eta = 0 \text{ for all leaves } L \in \mathcal{F}\} = \alpha \wedge \Omega^{\bullet-1}(M),$$

using Corollary 6.1.7 in the last equality. And $\overline{d}$ coincides with the leafwise de Rham differential $d_{\mathcal{F}}$, as by definition $\overline{d} \circ j = j \circ d$. Hence,

$$\left(\frac{\Omega^\bullet(M)}{\alpha \wedge \Omega^{\bullet-1}(M)}, \overline{d}\right) = \big(\Gamma(\wedge^\bullet D^*), d_{\mathcal{F}}\big).$$

Remark 6.1.9. We make some observations concerning (6.3).

i) Although for a fixed choice of α, the form β in (6.3) is not unique, the projection $j(\beta)$ is. Here j is the map defined in the sequence (6.4). Indeed, if $d\alpha = \beta \wedge \alpha = \beta' \wedge \alpha$, then $(\beta - \beta') \wedge \alpha = 0$, so that $\beta - \beta' \in \alpha \wedge \Omega^0(M) = C^\infty(M)\alpha$ by Lemma 6.1.6. Then $j(\beta - \beta') = 0$, so $j(\beta) = j(\beta')$.

[2]Alternatively, Corollary 6.1.7 implies that the complex $\alpha \wedge \Omega^{\bullet-1}(M)$ consists of exactly those differential forms whose pullback to the leaves vanishes. From this, it is immediate that $\alpha \wedge \Omega^{\bullet-1}(M)$ is a subcomplex of $\Omega^\bullet(M)$ (since the exterior derivative commutes with pullbacks) and that it is independent of the choice of α.

ii) If $d\alpha = \beta \wedge \alpha$, then $j(\beta)$ is closed for the differential $\overline{d}$. Indeed, we have

$$0 = d(d\alpha) = d(\beta \wedge \alpha) = (d\beta) \wedge \alpha - \beta \wedge (d\alpha) = (d\beta) \wedge \alpha - \beta \wedge \beta \wedge \alpha = (d\beta) \wedge \alpha,$$

so that $d\beta \in \alpha \wedge \Omega^1(M)$ by Lemma 6.1.6. This then implies that

$$\overline{d}(j(\beta)) = j(d\beta) = 0.$$

By the previous remark, we can now define the first foliation invariant.

Definition 6.1.10. Let $\mathcal{F}$ be a transversely orientable codimension-one foliation of M, with defining one-form $\alpha \in \Omega^1(M)$. The first obstruction class $\mathcal{C}_{\mathcal{F}}$ of $\mathcal{F}$ is a class in the first foliated cohomology group, defined as

$$\mathcal{C}_{\mathcal{F}} = [j(\beta)] \in H^1\left(\frac{\Omega^\bullet(M)}{\alpha \wedge \Omega^{\bullet-1}(M)}\right),$$

where $d\alpha = \beta \wedge \alpha$.

Let us check that $\mathcal{C}_{\mathcal{F}}$ only depends on the foliation $\mathcal{F}$, not on the choice of defining one-form. First off, as argued before, the complex $\Omega^\bullet(M)/\alpha\wedge\Omega^{\bullet-1}(M)$ is independent of choice of defining one-form. Next, if α and α' are defining one-forms for $\mathcal{F}$, then $\alpha' = f\alpha$ for some non-vanishing $f \in C^\infty(M)$. We have

$$d\alpha' = df \wedge \alpha + f d\alpha = df \wedge \alpha + f\beta \wedge \alpha = \left(\frac{df}{f} + \beta\right) \wedge \alpha',$$

so that $\beta' = d\log(|f|) + \beta$. Hence

$$j(\beta') = j(d\log(|f|)) + j(\beta) = \overline{d}(j(\log(|f|))) + j(\beta),$$

which implies that $[j(\beta')] = [j(\beta)]$.

The first obstruction class $\mathcal{C}_{\mathcal{F}}$ measures the obstruction to the existence of a closed defining one-form for $\mathcal{F}$.

Proposition 6.1.11. *Let $\mathcal{F}$ be a transversely orientable codimension-one foliation of M. The first obstruction class $\mathcal{C}_{\mathcal{F}}$ vanishes identically if and only if $\mathcal{F}$ has a closed defining one-form.*

Proof. We have the following equivalences:

$$\begin{aligned} [j(\beta)] = 0 &\Leftrightarrow j(\beta) = \overline{d}(j(f)) = j(df) && (f \in C^\infty(M)) \\ &\Leftrightarrow \beta - df \in \alpha \wedge \Omega^0(M) \\ &\Leftrightarrow \beta = df + g\alpha && (g \in C^\infty(M)). \end{aligned}$$

First assume that $\mathcal{C}_{\mathcal{F}} = 0$. Let α be a defining one-form for $\mathcal{F}$ such that $d\alpha = \beta \wedge \alpha$ with $\beta = df + g\alpha$ for some $f, g \in C^\infty(M)$. We consider $\alpha' := e^{-f}\alpha$, which is also a defining one-form for $\mathcal{F}$ since e^{-f} is non-vanishing. Moreover,

$$\begin{aligned} d\alpha' &= d\left(e^{-f}\alpha\right) \\ &= -e^{-f}df \wedge \alpha + e^{-f}d\alpha \\ &= -e^{-f}df \wedge \alpha + e^{-f}\beta \wedge \alpha \\ &= -e^{-f}df \wedge \alpha + e^{-f}(df + g\alpha) \wedge \alpha \end{aligned}$$

$$= -e^{-f}df \wedge \alpha + e^{-f}df \wedge \alpha$$
$$= 0.$$

Hence if $\mathcal{C}_\mathcal{F} = 0$, then we can find a closed defining one-form for $\mathcal{F}$. Conversely, if α is a closed defining one-form for $\mathcal{F}$, then $d\alpha = 0 = 0 \wedge \alpha$, so that we can take $\beta = 0$. It follows that

$$\mathcal{C}_\mathcal{F} = [j(0)] = 0.$$

$\square$

6.1.2 The second obstruction class

In what follows, we assume that M is endowed with a regular corank-one Poisson structure Π and that $\mathcal{F}$ is the corresponding codimension-one symplectic foliation. Furthermore, we assume that the first obstruction class $\mathcal{C}_\mathcal{F}$ vanishes, so that $\mathcal{F}$ is defined by a closed one-form $\alpha \in \Omega^1(M)$. We fix α throughout.

Definition 6.1.12. A two-form $\omega \in \Omega^2(M)$ is a defining two-form of the foliation $\mathcal{F}$ induced by the Poisson structure Π if $i_L^*\omega = \omega_L$ is the symplectic form on each leaf $i_L : L \hookrightarrow M$.

Remark 6.1.13. A codimension-one symplectic foliation $\mathcal{F}$ always has a defining two-form. This follows from exactness of the sequence (5.2), or merely from surjectivity of the map r in that sequence. In case $\mathcal{F}$ is induced by an orientable log-symplectic structure Λ on some ambient manifold, then one can choose a defining two-form ω such that $\iota_{X_\Lambda^\Omega|_M}\omega = 0$, by Theorem 5.1.1. With this extra condition, the defining two-form is unique (for fixed choice of volume form Ω).

Note that

$$i_L^*(d\omega) = d(i_L^*\omega) = d\omega_L = 0$$

for all leaves $L \in \mathcal{F}$, so that by Corollary 6.1.7 we can write

$$d\omega = \mu \wedge \alpha \quad \text{for some } \mu \in \Omega^2(M). \tag{6.5}$$

Remark 6.1.14. We make some observations concerning (6.5).

i) Although for a fixed choice of α, the form μ in (6.5) is not unique, the projection $j(\mu)$ is. Indeed, if $d\omega = \mu \wedge \alpha = \mu' \wedge \alpha$, then $(\mu - \mu') \wedge \alpha = 0$, so that $\mu - \mu' \in \alpha \wedge \Omega^1(M)$ by Lemma 6.1.6. Then $j(\mu - \mu') = 0$, so that $j(\mu) = j(\mu')$.

ii) If $d\omega = \mu \wedge \alpha$, then $j(\mu)$ is closed for the differential $\overline{d}$. Indeed, using the fact that α is closed, we have

$$0 = d(d\omega) = d(\mu \wedge \alpha) = (d\mu) \wedge \alpha + \mu \wedge (d\alpha) = (d\mu) \wedge \alpha,$$

so that $d\mu \in \alpha \wedge \Omega^2(M)$ by Lemma 6.1.6. This then implies that

$$\overline{d}(j(\mu)) = j(d\mu) = 0.$$

These observations allow us to define the second foliation invariant.

Definition 6.1.15. Let Π be a corank-one Poisson structure on M and let $\mathcal{F}$ be the induced codimension-one symplectic foliation. Assume that the first obstruction class $\mathcal{C}_\mathcal{F}$ vanishes, and fix a closed defining one-form α for $\mathcal{F}$. The second obstruction class $\sigma_\mathcal{F}$ of $\mathcal{F}$ is a class in the second foliated cohomology group, defined as

$$\sigma_\mathcal{F} = [j(\mu)] \in H^2\left(\frac{\Omega^\bullet(M)}{\alpha \wedge \Omega^{\bullet-1}(M)}\right),$$

where $d\omega = \mu \wedge \alpha$.

Let us check that the second obstruction class $\sigma_{\mathcal{F}}$ does not depend on the choice of defining two-form for $\mathcal{F}$. If ω and ω' are both defining two-forms for $\mathcal{F}$, then $i_L^*(\omega' - \omega) = \omega_L - \omega_L = 0$ for each leaf $L \in \mathcal{F}$. Hence by Corollary 6.1.7, we get

$$\omega' = \omega + \alpha \wedge \xi \quad \text{for some } \xi \in \Omega^1(M).$$

Therefore,

$$\begin{aligned} d\omega' &= d\omega + d\alpha \wedge \xi - \alpha \wedge d\xi \\ &= \mu \wedge \alpha - \alpha \wedge d\xi \\ &= (\mu - d\xi) \wedge \alpha, \end{aligned}$$

so that $\mu' = \mu - d\xi$. Hence

$$j(\mu') = j(\mu) - j(d\xi) = j(\mu) - \overline{d}(j(\xi)),$$

which implies that $[j(\mu')] = [j(\mu)]$.

The second obstruction class $\sigma_{\mathcal{F}}$ measures the obstruction to the existence of a closed defining two-form for $\mathcal{F}$.

Proposition 6.1.16. *Let Π be a corank-one Poisson structure on M and let $\mathcal{F}$ be the induced codimension-one symplectic foliation. Assume that the first obstruction class $\mathcal{C}_{\mathcal{F}}$ vanishes, and fix a closed defining one-form α for $\mathcal{F}$. The second obstruction class $\sigma_{\mathcal{F}}$ vanishes identically if and only if $\mathcal{F}$ has a closed defining two-form.*

Proof. We have the following equivalences:

$$\begin{aligned} [j(\mu)] = 0 &\Leftrightarrow j(\mu) = \overline{d}(j(\eta)) = j(d\eta) && (\eta \in \Omega^1(M)) \\ &\Leftrightarrow \mu - d\eta \in \alpha \wedge \Omega^1(M) \\ &\Leftrightarrow \mu = d\eta + \gamma \wedge \alpha && (\gamma \in \Omega^1(M)). \end{aligned}$$

First assume that $\sigma_{\mathcal{F}} = 0$. Let ω be a defining two-form for $\mathcal{F}$ with $d\omega = \mu \wedge \alpha$ and $\mu = d\eta + \gamma \wedge \alpha$ for some $\eta, \gamma \in \Omega^1(M)$. Consider $\omega' := \omega - \eta \wedge \alpha$. Then ω' is still a defining two-form for $\mathcal{F}$ since for each leaf $L \in \mathcal{F}$:

$$\begin{aligned} i_L^*\omega' &= i_L^*\omega - (i_L^*\eta) \wedge (i_L^*\alpha) \\ &= \omega_L - (i_L^*\eta) \wedge 0 \\ &= \omega_L. \end{aligned}$$

And ω' is closed since

$$\begin{aligned} d\omega' &= d\omega - (d\eta) \wedge \alpha + \eta \wedge d\alpha \\ &= \mu \wedge \alpha - (d\eta) \wedge \alpha \\ &= (d\eta + \gamma \wedge \alpha) \wedge \alpha - (d\eta) \wedge \alpha \\ &= (d\eta) \wedge \alpha - (d\eta) \wedge \alpha \\ &= 0. \end{aligned}$$

Hence if $\sigma_{\mathcal{F}} = 0$, then we can find a closed defining two-form for $\mathcal{F}$. Conversely, if ω is a closed defining two-form for $\mathcal{F}$, then $d\omega = 0 = 0 \wedge \alpha$, so that we can take $\mu = 0$. It follows that

$$\sigma_{\mathcal{F}} = [j(0)] = 0.$$

□

Remark 6.1.17. We can now reformulate the results obtained in Section 5.3.1 as follows: A corank-one Poisson structure (Z, Π_Z) is induced by a log-symplectic structure (on some ambient manifold M) if and only if $\mathcal{C}_{\mathcal{F}} = \sigma_{\mathcal{F}} = 0$, where $\mathcal{F}$ is the symplectic foliation of (Z, Π_Z).

Throughout this section, as well as in what follows, we only consider codimension-one symplectic foliations that are transversely orientable (i.e. defined by a one-form). We may wonder how stringent the assumption of transverse orientability is. In fact, if $\mathcal{F}$ is a codimension-one symplectic foliation of M, then the leafwise-symplectic forms induce an orientation on $T\mathcal{F}$, so that transverse orientability of $\mathcal{F}$ is equivalent with orientability of M.

6.2 Vanishing first invariant: unimodularity

Recall that to an orientable Poisson structure (M, Π) one can canonically associate its modular class $[X_\Pi] \in H^1_\Pi(M)$, which is the cohomology class of any modular vector field on M. The Poisson structure (M, Π) is called unimodular if this cohomology class $[X_\Pi]$ is zero. We will now show that unimodularity of (M, Π) is closely related with the first invariant $\mathcal{C}_{\mathcal{F}}$ of its symplectic foliation.

Let (M^{2n+1}, Π) be an orientable corank-one Poisson manifold and let $\mathcal{F}$ be its symplectic foliation. We can choose a defining one-form $\alpha \in \Omega^1(M)$ and a defining two-form $\omega \in \Omega^2(M)$ of $\mathcal{F}$. Then $\Theta := \alpha \wedge \omega^n$ is a volume form on M, as noted in (4.17). Let us compute the modular vector field of Π associated with this volume form. We calculate:

$$\begin{aligned}\iota_{X_f}(\alpha \wedge \omega^n) &= \iota_{X_f}(\alpha) \wedge \omega^n - \alpha \wedge \iota_{X_f}(\omega^n)\\ &= -\alpha \wedge \iota_{X_f}(\omega^n)\\ &= -n\alpha \wedge \iota_{X_f}(\omega) \wedge \omega^{n-1},\end{aligned}$$

using in the second equality that X_f is tangent to the leaves and therefore $\iota_{X_f}(\alpha) = 0$. Next, note that for any leaf $i_L : L \hookrightarrow M$, we have

$$\begin{aligned}i_L^*\left(\omega^\flat(X_f)\right) &= \omega_L^\flat\left(X_f|_L\right)\\ &= \omega_L^\flat\left((\Pi^\sharp(df))|_L\right)\\ &= \omega_L^\flat\left(\Pi_L^\sharp(i_L^*(df))\right)\\ &= -i_L^*(df),\end{aligned}$$

where $\omega_L = -\Pi_L^{-1}$ is the symplectic form on the leaf L. Hence we have for each leaf $L \in \mathcal{F}$ that

$$i_L^*\left(\omega^\flat(X_f) + df\right) = 0,$$

so that by Corollary 6.1.7,

$$\alpha \wedge \left(\omega^\flat(X_f) + df\right) = 0.$$

Hence

$$\alpha \wedge \omega^\flat(X_f) = -\alpha \wedge df,$$

so that altogether

$$\begin{aligned}\iota_{X_f}(\alpha \wedge \omega^n) &= -n\alpha \wedge \iota_{X_f}(\omega) \wedge \omega^{n-1}\\ &= n\alpha \wedge df \wedge \omega^{n-1}.\end{aligned}$$

Therefore, we get by using Cartan's magic formula:

$$\begin{aligned}\pounds_{X_f}\Theta &= d\left(\iota_{X_f}(\alpha\wedge\omega^n)\right)\\ &= nd(\alpha\wedge df\wedge\omega^{n-1})\\ &= n(d\alpha)\wedge df\wedge\omega^{n-1} - n\alpha\wedge d(df\wedge\omega^{n-1})\\ &= n\beta\wedge\alpha\wedge df\wedge\omega^{n-1}.\end{aligned}$$

In the last equality, we used (6.3) and also (6.5), which implies that $\alpha\wedge d\omega = 0$. On the other hand, we have

$$\pounds_{X_f}\Theta = X_\Pi^\Theta(f)\Theta = df(X_\Pi^\Theta)\alpha\wedge\omega^n = \left(\iota_{X_\Pi^\Theta}df\right)\alpha\wedge\omega^n,$$

so that

$$n\alpha\wedge df\wedge\beta\wedge\omega^{n-1} = \left(\iota_{X_\Pi^\Theta}df\right)\alpha\wedge\omega^n.$$

Now, the equation

$$\alpha\wedge\left(ndf\wedge\beta\wedge\omega^{n-1}\right) = \alpha\wedge\left(\left(\iota_{X_\Pi^\Theta}df\right)\omega^n\right)$$

implies

$$\alpha\wedge\left(ndf\wedge\beta\wedge\omega^{n-1} - \left(\iota_{X_\Pi^\Theta}df\right)\omega^n\right) = 0,$$

so that by Lemma 6.1.6 and Corollary 6.1.7

$$i_L^*\left(ndf\wedge\beta\wedge\omega^{n-1} - \left(\iota_{X_\Pi^\Theta}df\right)\omega^n\right) = 0$$

for each leaf $L\in\mathcal{F}$. Now

$$i_L^*\left(ndf\wedge\beta\wedge\omega^{n-1}\right) = ndf_L\wedge\beta_L\wedge\omega_L^{n-1},$$

where $\omega_L = i_L^*\omega, f_L = i_L^*f$ and $\beta_L = i_L^*\beta$. On the other hand, we have that X_Π^Θ is tangent to the leaves of $\mathcal{F}$ by Lemma 8.7.1 in the appendix. Using this fact, we have

$$\begin{aligned}i_L^*\left(\left(\iota_{X_\Pi^\Theta}df\right)\omega^n\right) &= i_L^*\left(\iota_{X_\Pi^\Theta}df\right)\omega_L^n\\ &= \left(\iota_{X_\Pi^\Theta}df\right)\Big|_L\omega_L^n\\ &= \iota_{X_\Pi^\Theta|_L}\left(i_L^*(df)\right)\omega_L^n\\ &= \left(\iota_{X_\Pi^\Theta|_L}df_L\right)\omega_L^n.\end{aligned}$$

So on each leaf $L\in\mathcal{F}$, we have

$$ndf_L\wedge\beta_L\wedge\omega_L^{n-1} = \left(\iota_{X_\Pi^\Theta|_L}df_L\right)\omega_L^n. \tag{6.6}$$

Because $df_L\wedge\omega_L^n$ is a $(2n+1)$-form on the $2n$-dimensional manifold L, we have

$$\iota_{X_\Pi^\Theta|_L}(df_L\wedge\omega_L^n) = 0,$$

which implies that

$$\left(\iota_{X_\Pi^\Theta|_L}df_L\right)\omega_L^n = df_L\wedge\left(n\iota_{X_\Pi^\Theta|_L}\omega_L\right)\wedge\omega_L^{n-1}. \tag{6.7}$$

By (6.6) and (6.7), we get

$$df_L\wedge\omega_L^{n-1}\wedge\left(\iota_{X_\Pi^\Theta|_L}\omega_L - \beta_L\right) = 0. \tag{6.8}$$

Since we started with arbitrary $f \in C^\infty(M)$ and the map $i_L^* : C^\infty(M) \to C^\infty(L)$ is surjective, it follows that (6.8) holds for all functions $f_L \in C^\infty(L)$. This implies that[3]

$$\omega_L^{n-1} \wedge \left(\iota_{X_\Pi^\Theta|_L} \omega_L - \beta_L \right) = 0.$$

Lemma 6.2.1 below then implies that

$$\iota_{X_\Pi^\Theta|_L} \omega_L = \beta_L.$$

Lemma 6.2.1. *Let (M^{2n}, ω) be a symplectic manifold. Then the map*

$$\Phi : \Omega^1(M) \to \Omega^{2n-1}(M) : \mu \mapsto \omega^{n-1} \wedge \mu$$

is an injective map of $C^\infty(M)$-modules.

Proof. Suppose that $\omega^{n-1} \wedge \mu = 0$. Pick $x \in M$ and let $(q_1, p_1, \ldots, q_n, p_n)$ be Darboux coordinates around x. We may then write locally around x:

$$\omega = \sum_{i=1}^n dq_i \wedge dp_i \quad \text{and} \quad \mu = \sum_{i=1}^n f_i dq_i + \sum_{j=1}^n g_j dp_j,$$

so that

$$\omega^{n-1} = (n-1)! \sum_{i=1}^n dq_1 \wedge dp_1 \wedge \cdots \wedge \widehat{dq_i \wedge dp_i} \wedge \cdots \wedge dq_n \wedge dp_n.$$

Then by assumption

$$0 = \left(\sum_{i=1}^n dq_1 \wedge dp_1 \wedge \cdots \wedge \widehat{dq_i \wedge dp_i} \wedge \cdots \wedge dq_n \wedge dp_n \right) \wedge \left(\sum_{i=1}^n f_i dq_i + \sum_{j=1}^n g_j dp_j \right),$$

which implies that

$$\sum_{i=1}^n (f_i \epsilon_i dq_1 \wedge \cdots \wedge \widehat{dp_i} \wedge \cdots \wedge dp_n + g_i \epsilon_i' dq_1 \wedge \cdots \wedge \widehat{dq_i} \wedge \cdots \wedge dp_n) = 0,$$

where $\epsilon_i, \epsilon_i' \in \{\pm 1\}$. Then one necessarily has $f_i = g_i = 0$ for $i = 1, \ldots, n$. $\square$

We have now proved the following:

[3]Indeed, let $\xi \in \Omega^{2n-1}(L)$ and assume that $df \wedge \xi = 0$ for all $f \in C^\infty(L)$. Pick $p \in L$ and write in coordinates $(U, x_1, \ldots, x_{2n})$ around p:

$$\xi = \sum_{i=1}^{2n} \xi_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{2n}.$$

Let $g \in C^\infty(L)$ be a smooth bump function supported inside U such that $g \equiv 1$ near p. For the functions $gx_j \in C^\infty(L)$, we have by assumption

$$\begin{aligned} 0 = d_p(gx_j) \wedge \xi_p &= d_p x_j \wedge \left(\sum_{i=1}^{2n} \xi_i(p) d_p x_1 \wedge \cdots \wedge \widehat{d_p x_i} \wedge \cdots \wedge d_p x_{2n} \right) \\ &= \pm \xi_j(p) d_p x_1 \wedge \cdots \wedge d_p x_{2n}, \end{aligned}$$

which implies that $\xi_j(p) = 0$. Therefore, $\xi = 0$.

Proposition 6.2.2. *Let (M^{2n+1}, Π) be an orientable corank-one Poisson structure with symplectic foliation $\mathcal{F}$. Fix a defining one-form α and a defining two-form ω of $\mathcal{F}$. Then the modular vector field X_Π^Θ with respect to the volume form $\Theta = \alpha \wedge \omega^n$ is the vector field which on each symplectic leaf $L \in \mathcal{F}$ satisfies*

$$\iota_{X_\Pi^\Theta|_L} \omega_L = \beta_L, \tag{6.9}$$

where ω_L is the symplectic form on L, $\beta_L = i_L^*\beta$ and $d\alpha = \beta \wedge \alpha$.

This statement makes sense. Although for a fixed defining one-form α, the differential form β is not uniquely determined, its pullback β_L to a leaf $L \in \mathcal{F}$ is unique. This was noted in i) of Remark 6.1.9. Next, the property (6.9) indeed defines X_Π^Θ uniquely: there exists only one vector field satisying this property, on each leaf L given by

$$\left(\omega_L^\flat\right)^{-1}(\beta_L),$$

using non-degeneracy of ω_L. As a corollary of Proposition 6.2.2, we obtain the following criterion for unimodularity.

Corollary 6.2.3. *An orientable corank-one Poisson manifold (M^{2n+1}, Π) with induced symplectic foliation $\mathcal{F}$ is unimodular if and only if the first obstruction class $\mathcal{C}_\mathcal{F}$ vanishes identically.*

Proof. First assume that $\mathcal{C}_\mathcal{F} = 0$. Choose a closed defining one-form $\alpha \in \Omega^1(M)$ and a defining two-form $\omega \in \Omega^2(M)$ for $\mathcal{F}$. Then

$$0 = d\alpha = \beta \wedge \alpha,$$

which implies that $\beta_L := i_L^*\beta = 0$ for each leaf $L \in \mathcal{F}$ (use Lemma 6.1.6 and Corollary 6.1.7). By Proposition 6.2.2, we get that the modular vector field X_Π^Θ for the volume form $\Theta := \alpha \wedge \omega^n$ satisfies

$$X_\Pi^\Theta\big|_L = \left(\omega_L^\flat\right)^{-1}(\beta_L) = \left(\omega_L^\flat\right)^{-1}(0) = 0$$

for each leaf $L \in \mathcal{F}$. Hence $X_\Pi^\Theta = 0$, which implies that (M^{2n+1}, Π) is unimodular.
Conversely, assume that (M^{2n+1}, Π) is unimodular. Fix a defining one-form α and a defining two-form ω for $\mathcal{F}$. We then know that there exists $h \in C^\infty(M)$ such that

$$X_\Pi^{\alpha\wedge\omega^n} = X_h.$$

Proposition 2.9.6 implies that with respect to the volume form $\mu := e^h\alpha \wedge \omega^n := \alpha' \wedge \omega^n$, we have

$$X_\Pi^\mu = X_h - X_{\log(e^h)} = X_h - X_h = 0.$$

Note that $\alpha' = e^h\alpha$ is also a defining one-form for $\mathcal{F}$. Let $\beta' \in \Omega^1(M)$ be such that

$$d\alpha' = \beta' \wedge \alpha'.$$

Since by Proposition 6.2.2

$$0 = X_\Pi^\mu\big|_L = \left(\omega_L^\flat\right)^{-1}(\beta'_L),$$

we get that $i_L^*\beta' = \beta'_L = 0$ for each leaf $L \in \mathcal{F}$. By Corollary 6.1.7, we get $\beta' = f\alpha'$ for some $f \in C^\infty(M)$, so that

$$d\alpha' = \beta' \wedge \alpha' = f\alpha' \wedge \alpha' = 0.$$

So α' is a closed defining one-form for $\mathcal{F}$, which implies that $\mathcal{C}_\mathcal{F} = 0$ by Proposition 6.1.11. □

Remark 6.2.4. In particular, the singular locus of a log-symplectic manifold is a unimodular corank-one Poisson manifold.

We present an alternative argument to obtain Corollary 6.2.3. Let (M^{2n+1}, Π) be an orientable manifold with corank-one Poisson structure. Fix a defining one-form $\alpha \in \Omega^1(M)$ and a defining two-form $\omega \in \Omega^2(M)$ for the symplectic foliation $\mathcal{F}$ induced by Π. We consider the complex of differential forms along the leaves of $\mathcal{F}$, with projection map j, as in (6.4):

$$(\Omega^\bullet(M), d) \xrightarrow{j} \left(\frac{\Omega^\bullet(M)}{\alpha \wedge \Omega^{\bullet-1}(M)}, \overline{d} \right).$$

Lemma 6.2.5. *The map*

$$\overline{\Pi^\sharp} : \frac{\Omega^k(M)}{\alpha \wedge \Omega^{k-1}(M)} \to \mathfrak{X}^k(M) : j(\beta) \mapsto \Pi^\sharp(\beta) \tag{6.10}$$

is well-defined.

Proof. Assume $j(\beta) = j(\beta')$. Then $\beta' = \beta + \alpha \wedge \eta$ for some $\eta \in \Omega^{k-1}(M)$, so that

$$\Pi^\sharp(\beta') = \Pi^\sharp(\beta) + \Pi^\sharp(\alpha) \wedge \Pi^\sharp(\eta).$$

We argue that $\Pi^\sharp(\alpha) = 0$. Let $\gamma \in \Omega^1(M)$ be arbitrary. Since α vanishes on vector fields tangent to the leaves of $\mathcal{F}$, we have

$$0 = \langle \alpha, \Pi^\sharp(\gamma) \rangle = -\langle \Pi^\sharp(\alpha), \gamma \rangle.$$

So $\Pi^\sharp(\alpha)$ pairs to zero with any one-form on M, and therefore $\Pi^\sharp(\alpha) = 0$. It follows that $\Pi^\sharp(\beta') = \Pi^\sharp(\beta)$, so that the map $\overline{\Pi^\sharp}$ is well-defined. □

The maps (6.10) combine to a chain map (up to sign)

$$\overline{\Pi^\sharp} : \left(\frac{\Omega^\bullet(M)}{\alpha \wedge \Omega^{\bullet-1}(M)}, \overline{d} \right) \to (\mathfrak{X}^\bullet(M), d_\Pi).$$

Indeed, we have for $\eta \in \Omega^k(M)$:

$$-d_\Pi \left(\overline{\Pi^\sharp}(j(\eta)) \right) = -d_\Pi(\Pi^\sharp(\eta)) = \Pi^\sharp(d\eta) = \overline{\Pi^\sharp}(j(d\eta)) = \overline{\Pi^\sharp}(\overline{d}(j(\eta))),$$

using Lemma 2.8.4. So we get induced maps in cohomology

$$[\overline{\Pi^\sharp}] : H^k_{\mathcal{F}}(M) \to H^k_\Pi(M) : [j(\eta)] \mapsto [\Pi^\sharp(\eta)],$$

denoting the foliated cohomology groups by $H^\bullet_{\mathcal{F}}(M)$. In degree one, we have the following result, which is an observation of our own.

Lemma 6.2.6. *The linear map*

$$[\overline{\Pi^\sharp}] : H^1_{\mathcal{F}}(M) \to H^1_\Pi(M)$$

is injective and up to sign, it takes the first obstruction class $\mathcal{C}_{\mathcal{F}} \in H^1_{\mathcal{F}}(M)$ to the modular class $[X_\Pi] \in H^1_\Pi(M)$, that is

$$[\overline{\Pi^\sharp}](\mathcal{C}_{\mathcal{F}}) = -[X_\Pi]. \tag{6.11}$$

Proof. Writing $d\alpha = \beta \wedge \alpha$, we know that $\mathcal{C}_{\mathcal{F}} = [j(\beta)]$, so that $\mathcal{C}_{\mathcal{F}}$ is mapped to $[\Pi^\sharp(\beta)]$. We will show that

$$\Pi^\sharp(\beta) = -X_\Pi^{\alpha\wedge\omega^n}.$$

Since each leaf $L \in \mathcal{F}$ is a Poisson submanifold of M with induced Poisson structure Π_L, we have

$$\Pi^\sharp(\beta)\Big|_L = \Pi_L^\sharp(\beta_L) = -\left(\omega_L^\flat\right)^{-1}(\beta_L) = -\left. X_\Pi^{\alpha\wedge\omega^n}\right|_L,$$

using Proposition 6.2.2 in the last equality. This proves (6.11). As for injectivity of $[\overline{\Pi^\sharp}]$, assume that

$$0 = [\overline{\Pi^\sharp}]\left([j(\eta)]\right) = [\Pi^\sharp(\eta)].$$

Then $\Pi^\sharp(\eta)$ is a Hamiltonian vector field, so there exists $f \in C^\infty(M)$ such that $\Pi^\sharp(\eta) = \Pi^\sharp(df)$. This implies that on each leaf $L \in \mathcal{F}$:

$$\Pi_L^\sharp(i_L^*(\eta)) = \Pi_L^\sharp(i_L^*(df)),$$

and hence $i_L^*(\eta) = i_L^*(df)$ since $\Pi_L^\sharp$ is injective. So we get that for each leaf $L \in \mathcal{F}$:

$$i_L^*(\eta - df) = 0,$$

which implies that $j(\eta - df) = 0$ by Corollary 6.1.7. So $j(\eta) = j(df) = \overline{d}(j(f))$, which implies that

$$[j(\eta)] = [\overline{d}(j(f))] = 0.$$

□

A particular consequence of Lemma 6.2.6 is Corollary 6.2.3. Note that the map $[\overline{\Pi^\sharp}]$ in Lemma 6.2.6 is not surjective in general: it only reaches classes represented by Poisson vector fields that are tangent to the symplectic leaves. So classes with a representative that is transverse to the leaves at some point do not lie in the image of $[\overline{\Pi^\sharp}]$.

6.3 Vanishing first invariant: a stability theorem

We now prove a stability theorem for transversely orientable codimension-one foliations with vanishing first invariant on compact connected manifolds. It is similar to Reeb's global stability theorem[4].

Proposition 6.3.1. *Let $\mathcal{F}$ be a transversely orientable codimension-one foliation on a compact connected manifold M with $\mathcal{C}_{\mathcal{F}} = 0$. We then have:*

i) There exists a nontrivial family of diffeomorphisms $\Phi_t : M \to M$, defined for $t \in \mathbb{R}$, that takes leaves to leaves.

ii) If $\mathcal{F}$ contains a compact leaf L, then all leaves are compact.

iii) If $\mathcal{F}$ contains a compact leaf, then each leaf L of $\mathcal{F}$ has a saturated neighborhood U[5] and a projection $f : U \to I \subset \mathbb{R}$ such that the foliation $\mathcal{F}|_U$ is given by the fibers of f.

[4]Reeb's global stability theorem states the following: "Let $\mathcal{F}$ be a transversely orientable codimension-one foliation of a compact connected manifold M. If $\mathcal{F}$ contains a compact leaf L with finite fundamental group, then every leaf of $\mathcal{F}$ is diffeomorphic to L. Furthermore, M is the total space of a fibration $f : M \to S^1$ with fiber L, and $\mathcal{F}$ is the fiber foliation $\{f^{-1}(\theta) : \theta \in S^1\}$."

[5]A saturated neighborhood is a neighborhood that is a union of leaves.

Proof. i) Since $\mathcal{C}_{\mathcal{F}} = 0$, we can choose a closed defining one-form $\alpha \in \Omega^1(M)$ of $\mathcal{F}$. Note that α trivializes the conormal bundle $(TM/T\mathcal{F})^*$, so that also the normal bundle $TM/T\mathcal{F}$ is trivial. We can choose a global non-vanishing section v of $TM/T\mathcal{F}$, and rescaling v we can assume that $\alpha(v) = 1$. In particular, v is transverse to the foliation $\mathcal{F}$. Since M is compact, we have that v is complete, and we claim that its flow $\{\Phi_t : M \to M\}_{t\in\mathbb{R}}$ is the family of diffeomorphisms sought for. Note that

$$\pounds_v\alpha = d(\iota_v\alpha) + \iota_v d\alpha = 0,$$

which implies that $\Phi_t^*\alpha = \alpha$ for all $t \in \mathbb{R}$. Indeed, we have

$$\begin{aligned}\frac{d}{dt}(\Phi_t^*\alpha) &= \frac{d}{ds}\bigg|_{s=0}(\Phi_{t+s}^*\alpha) = \frac{d}{ds}\bigg|_{s=0}\left[(\Phi_s\circ\Phi_t)^*\,\alpha\right] = \frac{d}{ds}\bigg|_{s=0}\left[\Phi_t^*(\Phi_s^*\alpha)\right] \\ &= \Phi_t^*\left(\frac{d}{ds}\bigg|_{s=0}\Phi_s^*\alpha\right) = \Phi_t^*\left(\pounds_v\alpha\right) = 0,\end{aligned}$$

so that $\Phi_t^*\alpha$ is constant. But as $\Phi_0^*\alpha = \alpha$, this then gives $\Phi_t^*\alpha = \alpha$ for all $t \in \mathbb{R}$. We now show that each Φ_t takes leaves to leaves. So let $L \in \mathcal{F}$ be a leaf. We first show that $\Phi_t(L)$ is integral, i.e. that

$$T_{\Phi_t(p)}\Phi_t(L) = \operatorname{Ker}\left(\alpha_{\Phi_t(p)}\right) \qquad \text{for } p \in L. \tag{6.12}$$

If $w_2 \in T_{\Phi_t(p)}\Phi_t(L)$ then by surjectivity of $\left((\Phi_t)_*\right)_p$ there exists $w_1 \in T_pL$ such that $w_2 = \left((\Phi_t)_*\right)_p(w_1)$. We then have

$$\alpha_{\Phi_t(p)}(w_2) = \alpha_{\Phi_t(p)}\Big(\left((\Phi_t)_*\right)_p(w_1)\Big) = \left(\Phi_t^*\alpha\right)_p(w_1) = \alpha_p(w_1) = 0.$$

Hence we have the inclusion $T_{\Phi_t(p)}\Phi_t(L) \subset \operatorname{Ker}\left(\alpha_{\Phi_t(p)}\right)$, so that the equality (6.12) follows by counting dimensions. So $\Phi_t(L)$ is integral, and since the leaves of $\mathcal{F}$ are the maximal integral submanifolds of the distribution $\operatorname{Ker}(\alpha)$, there exists a leaf $L_t' \in \mathcal{F}$ such that $\Phi_t(L) \subset L_t'$. Composing with Φ_{-t} gives $L \subset \Phi_{-t}(L_t')$. But the same argument shows that $\Phi_{-t}(L_t')$ lies inside some leaf L_{-t}'. Hence

$$L \subset \Phi_{-t}(L_t') \subset L_{-t}',$$

so that $L = L_{-t}'$. In particular, $L = \Phi_{-t}(L_t')$, so that $\Phi_t(L) = L_t'$. So Φ_t takes leaves to leaves, and this finishes the first step.

ii) Let N be the union of all compact leaves in M. Then N is nonempty by assumption, and moreover N is open. Indeed, if L is a compact leaf, then we can find an open neighborhood

$$\{\Phi_t(L) : t \in (-\epsilon, \epsilon)\}$$

of L that is contained in N. But also $M \setminus N$ is open: let L' be a non-compact leaf and let $m \in L'$. Assume by contradiction that m would not be an interior point of $M \setminus N$. Take an open neighborhood

$$\{\Phi_t(L') : t \in (-\epsilon, \epsilon)\}$$

of m. Necessarily, this neighborhood then intersects a compact leaf L. So there exists $t_0 \in (-\epsilon, \epsilon)$ and $m' \in L'$ such that $\Phi_{t_0}(m') \in L$. But then the leaves $\Phi_{t_0}(L')$ and L intersect, so that $\Phi_{t_0}(L') = L$. Hence $L' = \Phi_{-t_0}(L)$ is compact, being the image of the compact set L under the continuous map Φ_{-t_0}. This is a contradiction. So m is an interior point of $M \setminus N$, showing that $M \setminus N$ is open. Since N is a nonempty clopen in M, and M is connected, it follows that $N = M$.

iii) Let $L \in \mathcal{F}$ be a leaf (which is automatically compact). Since α is closed and $i_L^*\alpha = 0$, the Relative Poincaré Lemma 1.3.9 gives a tubular neighborhood U of L and a function f on U such that $\alpha = df$ and $f|_L = 0$. Shrinking U if necessary, we may assume that U is saturated[6], and since L is connected, we can choose U to be connected as well. Note that f is a submersion, since $d_pf = \alpha_p \neq 0$ at all point $p \in U$. The leaves L' inside U satisfy

$$0 = i_{L'}^*\alpha = i_{L'}^*df = d(i_{L'}^*f) = d(f|_{L'}), \tag{6.13}$$

so that f is constant on each leaf. Since leaves are maximal with the property (6.13), it follows that the leaves inside U are the level sets of f. At last, since U is connected and f is an open map (being a submersion), we have that $f(U) := I \subset \mathbb{R}$ is an open interval. □

By compactness of M, we can patch together the local pieces of information found in iii) of Proposition 6.3.1 to obtain a global statement.

Proposition 6.3.2. *Let $\mathcal{F}$ be a transversely orientable codimension-one foliation on a compact connected manifold M with $\mathcal{C}_{\mathcal{F}} = 0$, and assume that $\mathcal{F}$ has a compact leaf. Then there exists a fiber bundle $F : M \to S^1$ such that $\mathcal{F}$ coincides with the fiber foliation of F.*

Proof. Let us first show that the leaf space $M/\mathcal{F}$ is a smooth manifold, when endowed with the quotient topology of $\pi : M \to M/\mathcal{F}$.

i) By Proposition 6.3.1, we know that all leaves of $\mathcal{F}$ are compact. The leaf space of every codimension-one foliation with compact leaves is Hausdorff [Eel, p. 364].

ii) The leaf space $M/\mathcal{F}$ is second countable. Indeed, the projection map $\pi : M \to M/\mathcal{F}$ is open [CN, p.47], and it is well-known that open quotients of second countable spaces are second countable.

iii) We now exhibit a smooth structure on $M/\mathcal{F}$. Choose $p \in M/\mathcal{F}$. We show that there exists an open neighborhood of p that is homeomorphic with an open subset of $\mathbb{R}$. Proposition 6.3.1 gives a saturated open U around $\pi^{-1}(p) = L$ and a submersion $f : U \to I \subset \mathbb{R}$ such that the leaves of $\mathcal{F}$ inside U are level sets of f. Then $\pi(U)$ is an open neighborhood of p and we define a map $\psi : \pi(U) \to I \subset \mathbb{R}$ by the following commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\;f\;} & I \\ {\scriptstyle \pi|_U}\big\downarrow & \nearrow_{\psi} & \\ \pi(U) & & \end{array} \tag{6.14}$$

That is, we define $\psi(L_q) = f(q)$, where $q \in U$ and L_q is the leaf through q. We claim that $\psi : \pi(U) \to I \subset \mathbb{R}$ is a homeomorphism. First note that ψ is injective: if $\psi(L_q) = \psi(L_r)$ then $f(q) = f(r)$ so that q and r lie in the same level set of f. That is, q and r belong to the same leaf, so $L_q = L_r$. Next, ψ is surjective: if $c \in I$ then there exists $q \in U$ with $f(q) = c$ by surjectivity of f, so that $\psi(\pi(q)) = f(q) = c$. Continuity of ψ is automatic: by the universal property of the quotient topology, we have that ψ is continuous if and only

[6] Here we use that all leaves of $\mathcal{F}$ are compact. Reeb proved in his thesis that the projection $\pi : M \to M/\mathcal{F}$ onto the leaf space is a closed map when $\mathcal{F}$ is a codimension-one foliation with compact leaves and $M/\mathcal{F}$ is endowed with the quotient topology [CO, p. 277]. Since the projection $\pi : M \to M/\mathcal{F}$ is closed, we may find a saturated open neighborhood W of L such that $L \subset W \subset U$. This is proved in [Kan, Proposition 1.2.6].

if $\psi \circ \pi|_U$ is continuous. But $\psi \circ \pi|_U = f$ is smooth, hence certainly continuous. At last, we check that ψ^{-1} is continuous. If V is open in $\pi(U)$, then we can write $V = O \cap \pi(U)$ where $O \subset M/\mathcal{F}$ is open. Then $\pi^{-1}(O)$ is open in M. Since U is saturated, we have,

$$\pi^{-1}(O \cap \pi(U)) = \pi^{-1}(O) \cap \pi^{-1}(\pi(U)) = \pi^{-1}(O) \cap U,$$

so that $\pi|_U^{-1}(V)$ is open in U. Since f is a submersion, it is an open map, and therefore

$$\psi(V) = f(\pi|_U^{-1}(V))$$

is open in I. We have now showed that $M/\mathcal{F}$ is covered by charts $(\pi(U), \psi)$

iv) It remains to show that these charts are smoothly compatible. Assume we are given charts $\varphi : \pi(U) \to J$ and $\psi : \pi(V) \to I$ such that $\pi(U) \cap \pi(V) \neq \emptyset$. We show that the map

$$\psi \circ \varphi^{-1} : \varphi(\pi(U) \cap \pi(V)) \to \psi(\pi(U) \cap \pi(V))$$

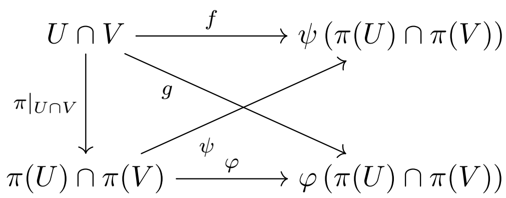
is smooth. Note that, since U and V are saturated, we have

$$\pi^{-1}\left(\pi(U) \cap \pi(V)\right) = \pi^{-1}(\pi(U)) \cap \pi^{-1}(\pi(V)) = U \cap V,$$

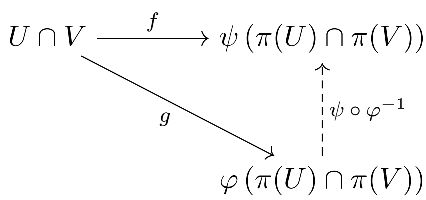
so that

$$\pi(U \cap V) = \pi\left(\pi^{-1}\left(\pi(U) \cap \pi(V)\right)\right) = \pi(U) \cap \pi(V),$$

using surjectivity of π in the last equality. Hence we get a commutative diagram



with surjective submersions $f : U \cap V \to \psi\left(\pi(U) \cap \pi(V)\right)$ and $g : U \cap V \to \varphi\left(\pi(U) \cap \pi(V)\right)$ as above. Inserting the map $\psi \circ \varphi^{-1}$ gives a commutative diagram



. (6.15)

Indeed, for all $p \in U \cap V$ we have

$$\left(\psi \circ \varphi^{-1}\right)(g(p)) = \left(\psi \circ \varphi^{-1}\right)(\varphi(\pi(p))) = \psi(\pi(p)) = f(p).$$

Since g is a smooth surjective submersion and f is smooth, also $\psi \circ \varphi^{-1}$ is smooth because of the commutative diagram (6.15). For a proof of this fact, see [Lee, Theorem 4.29].

We have now established that $M/\mathcal{F}$ is a smooth one-dimensional manifold. It is connected and compact, being the image of the connected compact space M under the continuous map $\pi : M \to M/\mathcal{F}$. Therefore $M/\mathcal{F}$ is diffeomorphic to S^1, and we obtain our candidate fibration $F : M \to S^1$ as the composition

$$M \xrightarrow{\pi} M/\mathcal{F} \xrightarrow{\sim} S^1.$$

The map π is smooth: its local representation in charts α on M and ψ on $M/\mathcal{F}$ is by 6.14

$$\psi \circ \pi \circ \alpha^{-1} = f \circ \alpha^{-1},$$

which is smooth. Hence also $F : M \to S^1$ is a smooth map. To see that $F : M \to S^1$ is indeed a fibration, it suffices to show that F is a proper surjective submersion, by Ehresmann's Lemma[7]. For sure F is surjective, being a composition of surjective maps. Next, we note that π is a submersion. Indeed, around $p \in M$ we find an open U, a submersion $f : U \to I \subset \mathbb{R}$ and a chart $\psi : \pi(U) \to I \subset \mathbb{R}$ such that $\psi \circ \pi|_U = f$ (see (6.14)). Taking derivatives, we get

$$d_p\left(\psi \circ \pi|_U\right) = d_{\pi(p)}\psi \circ d_p\pi = d_p f,$$

so that $d_p\pi$ is a composition of surjective maps

$$d_p\pi = d_{\pi(p)}\psi^{-1} \circ d_p f.$$

Hence π is a submersion, and therefore F is a submersion as well. At last, we check that $F : M \to S^1$ is proper. If $C \subset S^1$ is compact, then C is closed since S^1 is Hausdorff. By continuity of F, we get that $F^{-1}(C)$ is a closed subset of the compact space M, so that $F^{-1}(C)$ is compact as well. By Ehresmann's Lemma, $F : M \to S^1$ is indeed a fiber bundle, and the fibers of F clearly coincide with the leaves of $\mathcal{F}$. □

In particular, the leaves of $\mathcal{F}$ are all diffeomorphic, being the fibers of the fibration $F : M \to S^1$. We have now proved the following stability theorem:

Theorem 6.3.3. *Let $\mathcal{F}$ be a transversely orientable codimension-one foliation of a compact connected manifold M with $\mathcal{C}_\mathcal{F} = 0$. If $\mathcal{F}$ contains a compact leaf L, then every leaf of $\mathcal{F}$ is diffeomorphic to L. Furthermore, M is the total space of a fibration $F : M \to S^1$ with fiber L, and $\mathcal{F}$ is the fiber foliation $\{F^{-1}(\theta) : \theta \in S^1\}$.*

The condition that $\mathcal{C}_\mathcal{F} = 0$ is clearly necessary: if we have such a fibration $F : M \to S^1$, then $F^*(d\theta)$ is a closed defining one-form for the foliation $\mathcal{F}$, where θ is the "coordinate" on the circle.

We now further specify the fiber bundle structure $F : M \to S^1$ obtained in Theorem 6.3.3. The fibers of F are the leaves of $\mathcal{F}$, and in Proposition 6.3.1 we constructed a vector field v transverse to the leaves. Therefore, this vector field v defines an Ehresmann connection on M, at all points $p \in M$ given by

$$H_p := \mathbb{R}v_p.$$

We can then lift the loop in S^1 as follows: let $\gamma : [0,1] \to S^1$ be a parametrization of the circle. The tangent vector field γ' has a unique horizontal lift $(\gamma')^H$ to M. That is, $(\gamma')^H$ satisfies

$$\begin{cases} (\gamma')^H_p \in H_p & \text{for all } p \in M \\ (F_*)_p\left((\gamma')^H_p\right) = \gamma'_{F(p)} & \text{for all } p \in M \end{cases}.$$

Let ψ_p be the integral curve of $(\gamma')^H$, starting at $p \in F^{-1}(\gamma(0))$. By compactness of M, we have that ψ_p is defined for all time, and ψ_p lifts the loop in S^1 horizontally, namely

$$\begin{cases} \psi_p'(t) = (\gamma')^H_{\psi_p(t)} \in H_{\psi_p(t)} \\ F \circ \psi_p = \gamma \end{cases}.$$

[7] Ehresmann's Lemma states: "If $f : M \to N$ is a proper surjective submersion between smooth manifolds, then f is a locally trivial fibration." Recall that a map $f : X \to Y$ between topological spaces is said to be proper if inverse images of compact subsets of Y are compact in X.

Definition 6.3.4. In the above setup, let $L = F^{-1}(\gamma(0)) \in \mathcal{F}$. The holonomy map is the diffeomorphism ϕ defined by
$$\phi : L \to L : p \mapsto \psi_p(1),$$
where ψ_p is the integral curve of $(\gamma')^H$ starting at p.

We then obtain:

Corollary 6.3.5. *Let $\mathcal{F}$ be a transversely orientable codimension-one foliation of a compact connected manifold M with $\mathcal{C}_{\mathcal{F}} = 0$, and assume that the foliation contains a compact leaf L. Then the manifold M is the mapping torus*[8] *of the diffeomorphism $\phi : L \to L$ given by the holonomy map of the fibration over S^1.*

Proof. We are given fiber bundles $F : M \to S^1$ and $q : \frac{L\times[0,1]}{(x,0)\sim(\phi(x),1)} \to S^1$ with typical fiber L. A fiber bundle isomorphism between the two is given by

$$\begin{array}{ccc} \frac{L\times[0,1]}{(x,0)\sim(\phi(x),1)} & \xrightarrow{\quad\varphi\quad} & M \\ \Big\downarrow q & & \Big\downarrow F \\ \frac{[0,1]}{0\sim 1} & \xrightarrow[\bar{t} \mapsto \overline{1-t}]{} & \frac{[0,1]}{0\sim 1} \end{array},$$

where $\varphi(\overline{p,t}) = \psi_p(1-t)$ and ψ_p is the integral curve of $(\gamma')^H$ starting at p. Note that the map φ is well-defined since
$$\psi_p(1) = \phi(p) = \psi_{\phi(p)}(0).$$
□

6.4 Vanishing first and second invariant: a stability theorem

Now assume that (M^{2n+1}, Π) is an orientable corank-one Poisson structure, and let $\mathcal{F}$ be its symplectic foliation. We want to see what happens when both invariants $\mathcal{C}_{\mathcal{F}}$ and $\sigma_{\mathcal{F}}$ vanish. Recall from Proposition 6.1.11 and Proposition 6.1.16 that this is the case exactly when $\mathcal{F}$ has a closed defining one-form and a closed defining two-form.

Proposition 6.4.1. *Let (M^{2n+1}, Π) be an orientable corank-one Poisson structure with symplectic foliation $\mathcal{F}$. The invariants $\mathcal{C}_{\mathcal{F}}$ and $\sigma_{\mathcal{F}}$ vanish if and only if there exists a Poisson vector field transverse to the leaves of $\mathcal{F}$.*

Proof. This is the equivalence *ii)* $\Leftrightarrow$ *iii)* in Theorem 5.3.1. It is merely a consequence of Theorem 5.1.1. Concretely, given defining one- and two-forms α and ω, we consider the vector field v uniquely defined by
$$\begin{cases} \alpha(v) = 1 \\ \iota_v \omega = 0 \end{cases}. \tag{6.16}$$
Conversely, given a vector field v on M transverse to $\mathcal{F}$, we consider defining one- and two-forms α and ω uniquely specified by the equations (6.16). We showed in the proof of Theorem 5.1.1 that v is Poisson if and only if $d\alpha = d\omega = 0$. □

[8]The mapping torus of a diffeomorphism $\phi : L \to L$ is $\frac{L\times[0,1]}{(x,0)\sim(\phi(x),1)}$. For instance, if $L = (-\epsilon, \epsilon)$ and $\phi = -\mathrm{Id}$, then we obtain the Möbius strip.

Let (M^{2n+1}, Π) be an orientable corank-one Poisson structure with symplectic foliation $\mathcal{F}$. Assume moreover that M is compact and connected, that $\mathcal{C}_{\mathcal{F}} = \sigma_{\mathcal{F}} = 0$ and that $\mathcal{F}$ contains a compact leaf L. The conclusions of previous section remain valid, the only difference being that the transverse vector field v, the flow of which takes leaves to leaves, is now a Poisson vector field. The foliated manifold M again has a fiber bundle structure, with an Ehresmann connection defined by the vector field v. Since the parallel transport of the connection preserves the symplectic structure on the leaves, we obtain:

Theorem 6.4.2. *Let (M^{2n+1}, Π) be an orientable compact connected regular Poisson structure of corank one, and let $\mathcal{F}$ be its symplectic foliation. If $\mathcal{C}_{\mathcal{F}} = \sigma_{\mathcal{F}} = 0$ and $\mathcal{F}$ contains a compact leaf L, then every leaf of $\mathcal{F}$ is symplectomorphic to L. Furthermore, M is the total space of a fibration over S^1 and it is the mapping torus of the symplectomorphism $\phi : L \to L$ given by the holonomy map of the fibration over S^1.*

Proof. We only have to check that the flow Φ_t of the transverse Poisson vector field v above preserves the symplectic structures on the leaves. Suppose (L, ω) and (L', ω') are symplectic leaves of $\mathcal{F}$, and that $\Phi_t(L) = L'$. We have to show that $\left(\Phi_t|_L\right)^* (\omega') = \omega$. We first note that

$$\left(\Phi_t|_L\right)_* \Pi_L = \Pi_{L'},$$

where Π_L and $\Pi_{L'}$ are the Poisson structures on L and L' induced by Π. Indeed, since Φ_t is a Poisson diffeomorphism, we have $(\Phi_t)_* \Pi = \Pi$, so that restricting to L we get

$$\left(\Phi_t|_L\right)_* (\Pi_L) = \Pi|_{\Phi_t(L)} = \Pi_{L'}.$$

This implies that for all $p \in L$

$$\left(\Pi^\sharp_{L'}\right)_{\Phi_t(p)} = (d_p(\Phi_t|_L)) \circ \left(\Pi^\sharp_L\right)_p \circ (d_p(\Phi_t|_L))^*,$$

hence

$$\left((\omega')^\flat_{\Phi_t(p)}\right)^{-1} = (d_p(\Phi_t|_L)) \circ \left(\omega^\flat_p\right)^{-1} \circ (d_p(\Phi_t|_L))^*,$$

so that

$$(\omega')^\flat_{\Phi_t(p)} = \left((d_p(\Phi_t|_L))^{-1}\right)^* \circ \omega^\flat_p \circ (d_p(\Phi_t|_L))^{-1}.$$

It follows that for $v, w \in T_pL$, we have

$$\begin{aligned}
\left[\left(\Phi_t|_L\right)^* \omega'\right]_p (v, w) &= \omega'_{\Phi_t(p)} \left(d_p(\Phi_t|_L)(v), d_p(\Phi_t|_L)(w)\right) \\
&= \left(\omega'\right)^\flat_{\Phi_t(p)} \left(d_p(\Phi_t|_L)(v)\right) \left(d_p(\Phi_t|_L)(w)\right) \\
&= \left[\left((d_p(\Phi_t|_L))^{-1}\right)^* \circ \omega^\flat_p(v)\right] \left(d_p(\Phi_t|_L)(w)\right) \\
&= \left[\omega^\flat_p(v) \circ (d_p(\Phi_t|_L))^{-1}\right] \left(d_p(\Phi_t|_L)(w)\right) \\
&= \omega^\flat_p(v)(w) \\
&= \omega_p(v, w).
\end{aligned}$$

Hence $\Phi_t|_L : (L, \omega) \to (L', \omega')$ is a symplectomorphism. ☐

In particular, this theorem applies to the singular locus (Z, Π_Z) of a log-symplectic structure, provided that Z is compact and connected and that the foliation of Π_Z has a compact leaf.

Chapter 7

Outlook

In this chapter, we give a brief overview of some aspects of log-symplectic structures and related concepts that were not treated in detail in this thesis.

7.1 Deformations of log-symplectic structures

In [MO], one describes the space of Poisson bivectors near a given log-symplectic structure, up to small diffeomorphisms. Their main statement is the following:

Theorem 7.1.1. *Let (M, Z, Π) be a compact log-symplectic manifold. Consider $\overline{\omega_1}, \ldots, \overline{\omega_l}$ closed two-forms on M and $\gamma_1, \ldots, \gamma_k$ closed one-forms on Z such that their cohomology classes form a basis of $H^2(M)$ and of $H^1(Z)$ respectively. For $\epsilon \in \mathbb{R}^l$ and $\delta \in \mathbb{R}^k$, denote by $\overline{\omega_\epsilon} := \sum_{i=1}^l \epsilon_i \overline{\omega_i}$ and $\gamma_\delta := \sum_{i=1}^k \delta_i \gamma_i$. Then:*

i) For small enough $\epsilon \in \mathbb{R}^l$ and $\delta \in \mathbb{R}^k$, we have that the bivector $\Pi^{\overline{\omega_\epsilon}}_{\gamma_\delta}$ defined by

$$\left(\Pi^{\overline{\omega_\epsilon}}_{\gamma_\delta}\right)^{-1} := \Pi^{-1} + \overline{\omega_\epsilon} + d\log(\lambda) \wedge p^*\left(\gamma_\delta\right)$$

is a log-symplectic structure on M with singular locus Z. Here $p : E \to Z$ is a tubular neighborhood of Z, and λ is a distance function adapted to Z, as in Lemma 4.1.16.

ii) There is a C^1-open neighborhood $U \subset \Gamma(\wedge^2 TM)$ around Π, such that every Poisson structure $\Pi' \in U$ is isomorphic to $\Pi^{\overline{\omega_\epsilon}}_{\gamma_\delta}$ for some vectors $\epsilon \in \mathbb{R}^l, \delta \in \mathbb{R}^k$.

iii) There is a C^1-neighborhood $D \subset Diff(M)$ around Id_M such that for $\varphi \in D$, the equality $\varphi_*\left(\Pi^{\overline{\omega_\epsilon}}_{\gamma_\delta}\right) = \Pi^{\overline{\omega_{\epsilon'}}}_{\gamma_{\delta'}}$ implies $\epsilon = \epsilon'$ and $\delta = \delta'$.

Let us make some remarks on the different items in above theorem.

i) The given log-symplectic structure Π has an inverse b-symplectic form $\Pi^{-1} = \omega$, that can be decomposed as

$$\omega = \alpha + d\log(\lambda) \wedge p^*(\theta),$$

for closed differential forms $\alpha \in \Omega^2(M)$ and $\theta \in \Omega^1(Z)$. Adding a C^0-small closed b-two-form μ to ω yields another b-symplectic form, since non-degeneracy is an open condition. This implies part $i)$ of Theorem 7.1.1. If we decompose

$$\mu = \overline{\omega} + d\log(\lambda) \wedge p^*(\gamma),$$

for small closed differential forms $\overline{\omega} \in \Omega^2(M)$ and $\gamma \in \Omega^1(Z)$, then the deformed log-symplectic structure is the inverse of

$$\omega + \mu = \alpha + \overline{\omega} + d\log(\lambda) \wedge p^*(\theta + \gamma).$$

This deformation has the following geometric interpretation:

- On one hand, we deform Π by adding the restriction of $\overline{\omega}$ to the symplectic form on each leaf of Π. This type of transformation is called a Gauge transformation. It can be applied to any Poisson structure.
- On the other hand, we transform Π by changing the foliation on the singular locus Z: the foliation is no longer given by the kernel of θ, but it now integrates the kernel of $\theta + \gamma$. This type of transformation is specific to log-symplectic structures.

In fact, these two types of deformations cover all Poisson structures C^1-close to a log-symplectic structure, by *ii*) of Theorem 7.1.1.

ii) To have any hope at all that Poisson bivectors near a log-symplectic structure Π are again log-symplectic, one has to interpret "near" in the C^1-sense. If Π' is a C^0-small deformation of Π, then $\wedge^n \Pi'$ might very well no longer be transverse to the zero section of $\wedge^{2n} TM$. In order to preserve transversality, we need that the derivatives of Π (in a local trivialization) are not changed too much. That is, we have to consider C^1-deformations of Π.
To prove *ii*) in Theorem 7.1.1, one argues as follows. First one proves that a Poisson structure Π' that is C^1-close to the log-symplectic structure Π is also log-symplectic, possibly with different singular locus Z'. Then one finds a diffeomorphism $\psi(\Pi')$ that takes Π' to a log-symplectic structure with singular locus Z. If ω' is the b-symplectic form inverse to $\psi(\Pi')_*(\Pi')$, then it turns out that the class $[\omega'] \in {}^bH^2(M)$ is of the form $[\omega_{\epsilon,\delta}]$, where

$$\omega_{\epsilon,\delta} := \omega + \overline{\omega_\epsilon} + d\log(\lambda) \wedge p^*(\gamma_\delta).$$

One then obtains the conclusion *ii*) by applying a b-version of Moser's theorem[1].

Recall that by the b-Mazzeo-Melrose Theorem 4.3.1, we have that ${}^bH^2(M) \cong H^2(M) \oplus H^1(Z)$ via

$$[\omega] \mapsto ([\alpha], [\theta]),$$

where $\alpha + d\log(\lambda) \wedge p^*(\theta)$. Moreover, it is proved in [MO] that the pair $([\alpha], [\theta])$ is canonically associated with ω. So we can summarize Theorem 7.1.1 as follows: the Poisson structures C^1-close to Π are parameterized by an open neighborhood of 0 in ${}^bH^2(M) \cong H^2(M) \oplus H^1(Z)$, up to C^1-small diffeomorphisms.
This is in perfect analogy with the symplectic case: if (M, ω) is a compact symplectic manifold, then the space of symplectic structures C^0-near ω modulo diffeomorphisms connectable with the identity map corresponds to an open neighborhood of 0 in $H^2(M)$.
At last we recall that heuristically, the deformations of a Poisson structure Π are governed by the second Poisson cohomology group $H^2_\Pi(M)$. Theorem 7.1.1 makes this description very accurate in case Π is a log-symplectic structure, since we showed in Chapter 4 that $H^2_\Pi(M) \cong {}^bH^2(M)$ in that case.

[1] Lemma 2 in [MO]: "Let $\zeta \in {}^b\Omega^2(M)$ be b-symplectic form on a compact b-manifold (M, Z). If $\zeta' \in {}^b\Omega^2(M)$ is a closed b-two-form such that $(1-t)\zeta + t\zeta'$ is non-degenerate for all $t \in [0,1]$ and $[\zeta] = [\zeta'] \in {}^bH^2(M)$, then there exists a b-diffeomorphism $\varphi : (M, Z) \xrightarrow{\sim} (M, Z)$ such that $\varphi^*(\zeta') = \zeta$."

Example 7.1.2. Consider $S^2 \subset \mathbb{R}^3$, endowed with cylindrical coordinates θ and z. A log-symplectic structure on S^2 is given by

$$\Pi = z\frac{\partial}{\partial\theta} \wedge \frac{\partial}{\partial z},$$

as shown in Example 3.1.5. Its singular locus is the equator $S^1 \leftrightarrow \{z = 0\}$, and the inverse b-symplectic form is

$$\omega = \Pi^{-1} = \frac{dz}{z} \wedge d\theta.$$

Since the two-form $dz \wedge d\theta$ on S^2 is closed but not exact, its class $[dz \wedge d\theta]$ generates $H^2(S^2) \cong \mathbb{R}$. Similarly, the angular form $d\theta$ on S^1 is closed but not exact, so that its class $[d\theta]$ generates $H^1(S^1) \cong \mathbb{R}$. Theorem 7.1.1 now implies that every Poisson structure C^1-close to Π is isomorphic to one of the form

$$\begin{aligned}\Pi_{\epsilon,\delta} &:= \left(\frac{dz}{z} \wedge d\theta + \epsilon dz \wedge d\theta + \delta\frac{dz}{z} \wedge d\theta\right)^{-1} \\ &= \frac{z}{1 + \epsilon z + \delta}\frac{\partial}{\partial\theta} \wedge \frac{\partial}{\partial z}.\end{aligned}$$

7.2 Submanifold theory

Submanifold theory has been studied intensely in symplectic geometry. An important class of submanifolds consists of the Lagrangian ones, which are the half-dimensional submanifolds $N \subset (M, \omega)$ on which the symplectic form ω vanishes. Weinstein's well-known Lagrangian neighborhood theorem gives a normal form near a Lagrangian submanifold, namely:

Theorem 7.2.1 (Weinstein)**.** *If $L \subset (M, \omega)$ is a Lagrangian submanifold of a symplectic manifold (M, ω), then there exist a neighborhood U of L in M, a neighborhood V of L in T^*L and a symplectomorphism $f : (U, \omega) \to (V, \omega_{T^*L})$ that is the identity on L.*

Moreover, it is well-known that the graph of a one-form $\alpha \in \Gamma(T^*L) = \Omega^1(L)$ is a Lagrangian submanifold of T^*L if and only if α is closed. Therefore, the Lagrangian submanifolds near a given Lagrangian submanifold L correspond to small closed one-forms on L. One can show that the moduli space[2]

$$\frac{\{\text{Lagrangian submanifolds near } L\}}{\text{Hamiltonian diffeomorphisms}}$$

is an open subset of $H^1(L)$. Hence it is smooth and finite dimensional if L is compact.

Also for a Poisson manifold, there is a notion of Lagrangian submanifold C, defined by asking that for any symplectic leaf S of the Poisson manifold, $T_pC \cap T_pS$ is a Lagrangian subspace[3] of the symplectic vector space $\left(T_pS, (\omega_S)_p\right)$.

The submanifold theory of log-symplectic manifolds has not yet been addressed in the literature. One might be interested, for instance, in extending Weinstein's Lagrangian neighborhood theorem to the setting of log-symplectic manifolds. A related problem might be to describe the deformations of a Lagrangian submanifold L of a log-symplectic manifold, and to determine the moduli space of deformations.

[2]An isotopy $\{h_t\}$ is called Hamiltonian if there exists a smooth family of functions $H_t : M \to \mathbb{R}$ such that

$$\iota_{X_t}\omega = dH_t,$$

where $\{X_t\}$ is the time-dependent vector field associated with $\{h_t\}$. A Hamiltonian diffeomorphism is a symplectomorphism ϕ for which there exists a Hamiltonian isotopy $\{h_t\}$ such that $\phi = h_1$.

[3]That is, $\dim(T_pC \cap T_pS) = \frac{1}{2}\dim(T_pS)$ and $(\omega_S)_p\Big|_{(T_pC\cap T_pS)\times(T_pC\cap T_pS)} = 0$.

7.3 Generalizations

We defined log-symplectic structures as Poisson structures that degenerate linearly along a hypersurface. Various generalizations are possible by allowing more complicated degeneracies. For instance, the description of log-symplectic structures in terms of the b-tangent bundle immediately leads to a notion slightly more general than that of log-symplectic manifolds, in which the singular locus Z is no longer a smooth hypersurface. These structures also appear under the name "log-symplectic".

Definition 7.3.1. Fix a $2n$-dimensional manifold M, and let Z be a union of smooth hypersurfaces of normal crossing type[4] (i.e. locally there is a chart for which Z is a union of a collection of coordinate hyperplanes in $\mathbb{R}^{2n}$). Let $logZ$ denote the Lie algebroid whose sections are the vector fields on M tangent to Z. A Poisson tensor $\Pi \in \Gamma(\wedge^2 TM)$ is called log-symplectic if it is the image under the anchor map of a closed non-degenerate section of $\wedge^2(logZ)^*$.

In the same flavor, Lanius considered star log-symplectic structures, whose degeneracy loci are locally modeled by a finite set of lines in the plane intersecting at a point. She classified these structures on compact oriented manifolds in [Lan].

One can also consider higher order singularities. For a manifold M with specified hypersurface $Z \subset M$, we defined the b-tangent bundle to be the vector bundle whose sections are the vector fields on M that are tangent to Z. Similarly, one can define a vector bundle, called the b^k-tangent bundle, whose sections are vector fields with "order k tangency to Z", in some sense that is made precise in [Sco]. If Z is locally given by $y = 0$, then one has

$$^{b^k}T_pM = \begin{cases} T_pM & \text{if } p \notin Z \\ T_pZ + \left\langle y^k \frac{\partial}{\partial y} \right\rangle & \text{if } p \in Z \end{cases} \quad ; \qquad ^{b^k}T_p^*M = \begin{cases} T_p^*M & \text{if } p \notin Z \\ T_p^*Z + \left\langle \frac{dy}{y^k} \right\rangle & \text{if } p \in Z \end{cases} .$$

The sections of the exterior algebra of $^{b^k}T^*M$ are b^k-forms. They form a complex, the cohomology of which again allows a Mazzeo-Melrose type of decomposition theorem. A b^k-symplectic form is a closed b^k-two-form of full rank, and the classical theorems from symplectic geometry generalize further to the b^k-category. For instance, a b^k-version of Moser's theorem yields the b^k-Darboux theorem, which states that a b^k-symplectic form ω on (M, Z) locally looks like

$$\omega = \frac{dx_1}{x_1^k} \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i,$$

where Z is locally defined by $x_1 = 0$ [MP].

A Poisson structure is said to be of b^k-type if it is dual to a b^k-symplectic form. On a surface for instance, such Poisson structures are given by $f\Pi_0$, where Π_0 is dual to a symplectic form and f is locally the k-th power of a defining function for Z. Scott classified these Poisson structures of b^k-type on compact oriented surfaces in [Sco].

Of course, many aspects of log-symplectic structures still remain untreated in this thesis, like the associated symplectic groupoids, integrable systems... [GMP2].

[4] This concept appeared for instance in [GLPR, p. 4], under the name "normal crossing divisor".

Chapter 8

Appendices

8.1 On skew-symmetric bilinear maps

Proposition 8.1.1 (Standard form). *Let V be an n-dimensional real vector space, and let $\psi : V \times V \to \mathbb{R}$ be a skew-symmetric bilinear map. Then there exists a basis $\{v_1, \ldots, v_n\}$ with respect to which the matrix $\big[\psi(v_i, v_j)\big]_{i,j}$ has the form*

$$[\psi] = \left[\begin{array}{cccccccc|c} 0 & 1 & & & & & & & \\ -1 & 0 & & & & 0 & & & \\ & & 0 & 1 & & & & & \\ & & -1 & 0 & & & & & 0 \\ & & & & \ddots & & & & \\ & 0 & & & & 0 & 1 & & \\ & & & & & -1 & 0 & & \\ \hline & & & & 0 & & & & 0 \end{array}\right]$$

In particular, the rank of ψ is even.

Proof. ([Ca]) By induction on $n = \dim(V)$. If $\psi \equiv 0$, then we are done. Otherwise, there exist $v_1, v_2 \in V$ with $\psi(v_1, v_2) \neq 0$. Rescaling these vectors, we can assume that $\psi(v_1, v_2) = 1$.
Let $W := \operatorname{span}\{v_1, v_2\}$ and $W^\perp := \{v \in V : \ \psi(v, w) = 0 \text{ for all } w \in W\}$. Then $V = W \oplus W^\perp$:

- $W \cap W^\perp = \{0\}$: Suppose that $v = av_1 + bv_2 \in W \cap W^\perp$. Then

$$\begin{cases} 0 = \psi(v, v_1) = -b \\ 0 = \psi(v, v_2) = a \end{cases} \quad \Rightarrow v = 0.$$

- $V = W + W^\perp$: Suppose that $v \in V$ has $\psi(v, v_1) = c$ and $\psi(v, v_2) = d$. Then

$$v = (-cv_2 + dv_1) + (v + cv_2 - dv_1) \in W + W^\perp.$$

By the induction hypothesis, there exists a basis $\{v_3, \ldots, v_n\}$ of $W^\perp$ with respect to which $\psi|_{W^\perp}$ is represented by a matrix of the desired form. Since $\psi|_W$ has matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with respect to $\{v_1, v_2\}$, it follows that the basis $\{v_1, v_2, v_3, \ldots, v_n\}$ satisfies the criteria. □

Lemma 8.1.2. *Let V be a $2n$-dimensional real vector space, and $\psi : V \times V \to \mathbb{R}$ a skew-symmetric bilinear form. Then ψ is non-degenerate if and only if $\wedge^n \psi \neq 0$.*

Proof. Assume that ψ is non-degenerate. By Proposition 8.1.1, there exists a basis $\{e_1, f_1, \ldots, e_n, f_n\}$ of V with respect to which the matrix of ψ is

$$[\psi] = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}.$$

Then $\{e_i^* \wedge e_j^*,\ f_i^* \wedge f_j^*,\ e_a^* \wedge f_b^* :\ 1 \le i < j \le n,\ 1 \le a, b \le n\}$ is a basis of $\bigwedge^2 V^*$, with respect to which

$$\psi = \sum_{i=1}^n e_i^* \wedge f_i^*.$$

Indeed, on basis vectors of $\{e_1, f_1, \ldots, e_n, f_n\}$, we have

- $\left(\sum_{i=1}^n e_i^* \wedge f_i^*\right)(e_k, e_j) = \sum_{i=1}^n \begin{vmatrix} e_k(e_i^*) & e_k(f_i^*) \\ e_j(e_i^*) & e_j(f_i^*) \end{vmatrix} = 0 = \psi(e_k, e_j).$
- $\left(\sum_{i=1}^n e_i^* \wedge f_i^*\right)(f_k, f_j) = \sum_{i=1}^n \begin{vmatrix} f_k(e_i^*) & f_k(f_i^*) \\ f_j(e_i^*) & f_j(f_i^*) \end{vmatrix} = 0 = \psi(f_k, f_j).$
- $\left(\sum_{i=1}^n e_i^* \wedge f_i^*\right)(e_k, f_j) = \sum_{i=1}^n \begin{vmatrix} e_k(e_i^*) & e_k(f_i^*) \\ f_j(e_i^*) & f_j(f_i^*) \end{vmatrix} = \sum_{i=1}^n \delta_{ik}\delta_{ij} = \delta_{kj} = \psi(e_k, f_j).$

Hence

$$\begin{aligned} \wedge^n \psi = \wedge^n \left(\sum_{i=1}^n e_i^* \wedge f_i^* \right) &= \sum_{\sigma \in S_n} (e_{\sigma(1)}^* \wedge f_{\sigma(1)}^*) \wedge \cdots \wedge (e_{\sigma(n)}^* \wedge f_{\sigma(n)}^*) \\ &= \sum_{\sigma \in S_n} (e_1^* \wedge f_1^*) \wedge \cdots \wedge (e_n^* \wedge f_n^*) = n!\, e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^*. \end{aligned}$$

In particular,

$$\wedge^n \psi(e_1, f_1, \ldots, e_n, f_n) = n! \begin{vmatrix} e_1(e_1^*) & e_1(f_1^*) & \cdots & e_1(f_n^*) \\ f_1(e_1^*) & f_1(f_1^*) & \cdots & f_1(f_n^*) \\ \vdots & \vdots & \vdots & \vdots \\ f_n(e_1^*) & f_n(f_1^*) & \cdots & f_n(f_n^*) \end{vmatrix} = n! \neq 0.$$

If ψ is degenerate, then it has rank $2k$ for some $k < n$. Proposition 8.1.1 gives a basis $\{e_1, f_1, \ldots, e_k, f_k, \ldots, e_n, f_n\}$ of V with respect to which the matrix of ψ is

$$[\psi] = \left[\begin{array}{ccccc|c} 0 & 1 & & & & \\ -1 & 0 & & 0 & & \\ & & \ddots & & & 0 \\ & 0 & & 0 & 1 & \\ & & & -1 & 0 & \\ \hline & & 0 & & & 0_{(2n-2k)} \end{array}\right],$$

where $0_{(2n-2k)}$ is the zero matrix of dimensions $(2n-2k)\times(2n-2k)$. As earlier, $\psi = \sum_{i=1}^k e_i^* \wedge f_i^*$, and the graded symmetry of $\wedge$ implies that

$$\wedge^n \psi = \wedge^n \left(\sum_{i=1}^k e_i^* \wedge f_i^* \right) = 0.$$

$\square$

8.2 Calculus with differential forms

Lemma 8.2.1 (Cartan's magic formula)**.** *If $X \in \mathfrak{X}(M)$ is a vector field and $\omega \in \Omega^k(M)$ is a differential form, then*

$$\pounds_X \omega = d\iota_X \omega + \iota_X d\omega.$$

Proof. By induction on the degree k of ω.
If $f \in \Omega^0(M) = C^\infty(M)$, then $d\iota_X f + \iota_X df = \iota_X df = df(X) = \pounds_X f$.
Assume that the formula holds for $(k-1)$-forms. By linearity of the operators $\pounds_X$ and $d\iota_X + \iota_X d$, it is enough to prove the formula for $\omega = f dx_1 \wedge \cdots \wedge dx_k$. We write $\omega = dx_1 \wedge \omega_1$ for $\omega_1 := f dx_2 \wedge \cdots \wedge dx_k$. Since $\pounds_X$ is a degree zero derivation of $\wedge$, we get

$$\pounds_X \omega = \pounds_X (dx_1 \wedge \omega_1) = (\pounds_X dx_1) \wedge \omega_1 + dx_1 \wedge \pounds_X \omega_1.$$

On the other hand, since d and ι_X are degree 1 resp. -1 derivations of $\wedge$, we obtain

$$\begin{aligned} d\iota_X \omega + \iota_X d\omega &= d\iota_X (dx_1 \wedge \omega_1) + \iota_X d(dx_1 \wedge \omega_1) \\ &= d\left((\iota_X dx_1) \wedge \omega_1 - dx_1 \wedge \iota_X \omega_1\right) - \iota_X (dx_1 \wedge d\omega_1) \\ &= (d\iota_X dx_1) \wedge \omega_1 + (\iota_X dx_1) \wedge d\omega_1 + dx_1 \wedge d\iota_X \omega_1 - (\iota_X dx_1) \wedge d\omega_1 + dx_1 \wedge \iota_X d\omega_1 \\ &= (d\pounds_X x_1) \wedge \omega_1 + dx_1 \wedge \pounds_X \omega_1 \\ &= (\pounds_X dx_1) \wedge \omega_1 + dx_1 \wedge \pounds_X \omega_1, \end{aligned}$$

where the penultimate equality holds by the case $k = 0$ and the induction hypothesis. In the last equality, we used that $\pounds_X$ and d commute. $\square$

Let $\{\rho_t\}$ be an isotopy on a manifold M with corresponding time-dependent vector field $\{X_t\}$, that is

$$\frac{d}{dt}\rho_t = X_t \circ \rho_t. \tag{8.1}$$

Recall that the Lie derivative of a differential form $\omega \in \Omega^k(M)$ with respect to the time-dependent vector field $\{X_t\}$ is defined as

$$\pounds_{X_t} \omega = \left.\frac{d}{dt}\right|_{t=0} \rho_t^* \omega.$$

We have the following formula:

Lemma 8.2.2. *Let $\{\rho_t\}$ be an isotopy on a manifold M with corresponding time-dependent vector field $\{X_t\}$. Then*

$$\frac{d}{dt}\rho_t^* \omega = \rho_t^* \pounds_{X_t} \omega \qquad \text{for any } \omega \in \Omega(M).$$

Proof. ([Ler]) For fixed $t \in [0,1]$, we define operators Q_1 and Q_2 on $\Omega(M)$ by

$$Q_1(\omega) = \frac{d}{dt}\rho_t^*\omega,$$
$$Q_2(\omega) = \rho_t^*(\pounds_{X_t}\omega).$$

We have to show that $Q_1 = Q_2$. To do this, it is enough to carry out the following steps:

1. Show that Q_1 and Q_2 coincide on functions.
2. Check that Q_1 and Q_2 commute with d.
3. Show that $Q_i(\nu \wedge \mu) = Q_i(\nu) \wedge \rho_t^*\mu + (\rho_t^*\nu) \wedge Q_i(\mu)$ for $i = 1, 2$.

For 1., we take $f \in C^\infty(M)$ and $x \in M$. We compute

$$\frac{d}{dt}(\rho_t^* f)(x) = \frac{d}{dt}(f(\rho_t(x))) = \big\langle df(\rho_t(x)), X_t(\rho_t(x))\big\rangle = (\pounds_{X_t} f)(\rho_t(x)) = \rho_t^*(\pounds_{X_t} f)(x),$$

where the second equality holds by the chain rule and Equation (8.1).
As for 2., we note that $d \circ \rho_t^* = \rho_t^* \circ d$. Applying d/dt to both sides and using that d/dt commutes with d gives $Q_1 \circ d = d \circ Q_1$. Similarly, using that $d \circ \rho_t^* = \rho_t^* \circ d$ and that $d \circ \pounds_{X_t} = \pounds_{X_t} \circ d$, we get $Q_2 \circ d = d \circ Q_2$.
For 3., we first note that $\rho_t^*(\nu \wedge \mu) = (\rho_t^*\nu) \wedge (\rho_t^*\mu)$. Differentiating both sides with respect to t gives

$$\frac{d}{dt}\left(\rho_t^*(\nu \wedge \mu)\right) = \left(\frac{d}{dt}(\rho_t^*\nu)\right) \wedge (\rho_t^*\mu) + (\rho_t^*\nu) \wedge \left(\frac{d}{dt}(\rho_t^*\mu)\right),$$

which shows 3. for Q_1. Similarly, since $\pounds_{X_t}(\nu \wedge \mu) = (\pounds_{X_t}\nu) \wedge \mu + \nu \wedge (\pounds_{X_t}\mu)$, we get

$$\rho_t^*\left(\pounds_{X_t}(\nu \wedge \mu)\right) = (\rho_t^*\pounds_{X_t}\nu) \wedge (\rho_t^*\mu) + (\rho_t^*\nu) \wedge (\rho_t^*\pounds_{X_t}\mu),$$

which shows 3. for Q_2. □

We need the following improved version of Lemma 8.2.2.

Lemma 8.2.3. *Let $\{\rho_t\}$ be an isotopy on a manifold M with corresponding time-dependent vector field $\{X_t\}$. Then*

$$\frac{d}{dt}\rho_t^*\omega_t = \rho_t^*\left(\pounds_{X_t}\omega_t + \frac{d\omega_t}{dt}\right),$$

for any smooth family of k-forms ω_t.

Proof. ([Ca]) If $f(x, y)$ is a function of two variables, then we have by the chain rule

$$\frac{d}{dt}f(t,t) = \left.\frac{d}{dx}\right|_{x=t} f(x,t) + \left.\frac{d}{dy}\right|_{y=t} f(t,y).$$

Therefore,

$$\begin{aligned}
\frac{d}{dt}\rho_t^*\omega_t &= \left.\frac{d}{dx}\right|_{x=t} \rho_x^*\omega_t + \left.\frac{d}{dy}\right|_{y=t} \rho_t^*\omega_y \\
&= \rho_x^*\pounds_{X_x}\omega_t\big|_{x=t} + \rho_t^*\left.\frac{d\omega_y}{dy}\right|_{y=t} \\
&= \rho_t^*\left(\pounds_{X_t}\omega_t + \frac{d\omega_t}{dt}\right),
\end{aligned}$$

using Lemma 8.2.2 and linearity of ρ_t^*. □

8.3 Multivector fields

This lemma is Exercise 1.5 in Homework 1 of [FM].

Lemma 8.3.1. *If $f \in C^\infty(M)$ and $\nu \in \mathfrak{X}^k(M)$, then $[f,\nu] = -\iota_{df}\nu$.*

Proof. By induction on the degree k of ν.
If $g \in \mathfrak{X}^0(M) = C^\infty(M)$, then both $[f,g]$ and $\iota_{df}g$ lie in $\mathfrak{X}^{-1}(M) = \{0\}$. So $[f,g] = -\iota_{df}g$.
If $X \in \mathfrak{X}(M)$, then

$$[f,X] = -[X,f] = -\pounds_X f = -\langle df, X\rangle = -\iota_{df}X.$$

Assuming that the formula holds for k-vector fields, let $\nu \in \mathfrak{X}^{k+1}(M)$. In local coordinates, we have

$$\begin{aligned}
[f,\nu] &= \left[f, \sum_{i_1<\ldots<i_{k+1}} \nu_{i_1,\ldots,i_{k+1}} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{k+1}}}\right] = \sum_{i_1<\ldots<i_{k+1}} \left[f, \nu_{i_1,\ldots,i_{k+1}} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{k+1}}}\right] \\
&= \sum_{i_1<\ldots<i_{k+1}} [f,\nu_{i_1,\ldots,i_{k+1}}] \wedge \left(\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{k+1}}}\right) + \nu_{i_1,\ldots,i_{k+1}} \left[f, \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{k+1}}}\right] \\
&= \sum_{i_1<\ldots<i_{k+1}} \nu_{i_1,\ldots,i_{k+1}} \left[f, \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{k+1}}}\right] \quad \text{(by the case } k=0\text{)} \\
&= \sum_{i_1<\ldots<i_{k+1}} \nu_{i_1,\ldots,i_{k+1}} \left(\left[f, \frac{\partial}{\partial x_{i_1}}\right] \wedge \left(\frac{\partial}{\partial x_{i_2}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{k+1}}}\right) - \frac{\partial}{\partial x_{i_1}} \wedge \left[f, \frac{\partial}{\partial x_{i_2}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{k+1}}}\right]\right) \\
&= \sum_{i_1<\ldots<i_{k+1}} \nu_{i_1,\ldots,i_{k+1}} \left(-\iota_{df}\left(\frac{\partial}{\partial x_{i_1}}\right) \wedge \left(\frac{\partial}{\partial x_{i_2}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{k+1}}}\right) + \frac{\partial}{\partial x_{i_1}} \wedge \iota_{df}\left(\frac{\partial}{\partial x_{i_2}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{k+1}}}\right)\right) \\
&\quad \text{(by the case } k=1 \text{ and the induction hypothesis)} \\
&= \sum_{i_1<\ldots<i_{k+1}} \nu_{i_1,\ldots,i_{k+1}} \left(-\iota_{df}\left(\frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{k+1}}}\right)\right) \\
&= -\iota_{df}\left(\sum_{i_1<\ldots<i_{k+1}} \nu_{i_1,\ldots,i_{k+1}} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_{k+1}}}\right) = -\iota_{df}\nu.
\end{aligned}$$

□

Next lemma is Exercise 1.7 in Homework 1 of [FM].

Lemma 8.3.2. *Let $\Phi : M \to N$ be a smooth map. Let $\phi^1 \in \mathfrak{X}^k(M), \xi^1 \in \mathfrak{X}^l(M), \phi^2 \in \mathfrak{X}^k(N)$ and $\xi^2 \in \mathfrak{X}^l(N)$. If ϕ^1 and ϕ^2 are Φ-related, and ξ^1 and ξ^2 are Φ-related, then also $[\phi^1,\xi^1]$ and $[\phi^2,\xi^2]$ are Φ-related.*

Proof. Recall that to $\nu \in \mathfrak{X}^j(M)$, we associate a multilinear map

$$\overline{\nu} : C^\infty(M) \times \cdots \times C^\infty(M) \to C^\infty(M) : (f_1,\ldots,f_j) \mapsto \nu(df_1,\ldots,df_j).$$

We have

$$\begin{aligned}
\overline{\phi^1}(f_1\circ\Phi,\ldots,f_k\circ\Phi)(x) &= \phi^1_x\big(d_x(f_1\circ\Phi),\ldots,d_x(f_k\circ\Phi)\big) \\
&= \phi^1_x\big(d_{\Phi(x)}f_1\circ d_x\Phi,\ldots,d_{\Phi(x)}f_k\circ d_x\Phi\big) \\
&= \big[(d_x\Phi)\phi^1_x\big]\,(d_{\Phi(x)}f_1,\ldots,d_{\Phi(x)}f_k),
\end{aligned}$$

whereas

$$\left(\overline{\phi^2}(f_1,\ldots,f_k)\circ\Phi\right)(x)=\overline{\phi^2}(f_1,\ldots,f_k)(\Phi(x))=\phi^2_{\Phi(x)}(d_{\Phi(x)}f_1,\ldots,d_{\Phi(x)}f_k).$$

Since by assumption, $\phi^2_{\Phi(x)}=(d_x\Phi)\phi^1_x$, we get

$$\overline{\phi^1}(f_1\circ\Phi,\ldots,f_k\circ\Phi)=\overline{\phi^2}(f_1,\ldots,f_k)\circ\Phi, \tag{8.2}$$

and similarly

$$\overline{\xi^1}(f_1\circ\Phi,\ldots,f_k\circ\Phi)=\overline{\xi^2}(f_1,\ldots,f_k)\circ\Phi. \tag{8.3}$$

Next, using Equations (8.2) and (8.3), we get[1]

$$\begin{aligned}
&\left(\phi^2\circ\xi^2\right)_{\Phi(x)}\left(d_{\Phi(x)}f_1,\ldots,d_{\Phi(x)}f_{k+l-1}\right)\\
&\qquad=\sum_\sigma \operatorname{sgn}(\sigma)\overline{\phi^2}\left(\overline{\xi^2}(f_{\sigma(1)},\ldots,f_{\sigma(k)}),f_{\sigma(k+1)},\ldots,f_{\sigma(k+l-1)}\right)(\Phi(x))\\
&\qquad=\sum_\sigma \operatorname{sgn}(\sigma)\overline{\phi^1}\left(\overline{\xi^2}(f_{\sigma(1)},\ldots,f_{\sigma(k)})\circ\Phi,f_{\sigma(k+1)}\circ\Phi,\ldots,f_{\sigma(k+l-1)}\circ\Phi\right)(x)\\
&\qquad=\sum_\sigma \operatorname{sgn}(\sigma)\overline{\phi^1}\left(\overline{\xi^1}(f_{\sigma(1)}\circ\Phi,\ldots,f_{\sigma(k)}\circ\Phi),f_{\sigma(k+1)}\circ\Phi,\ldots,f_{\sigma(k+l-1)}\circ\Phi\right)(x)\\
&\qquad=\left(\phi^1\circ\xi^1\right)_x\left(d_x(f_1\circ\Phi),\ldots,d_x(f_{k+l-1}\circ\Phi)\right)\\
&\qquad=\left(\phi^1\circ\xi^1\right)_x\left(d_{\Phi(x)}f_1\circ d_x\Phi,\ldots,d_{\Phi(x)}f_{k+l-1}\circ d_x\Phi\right)\\
&\qquad=\left[(d_x\Phi)(\phi^1\circ\xi^1)_x\right]\left(d_{\Phi(x)}f_1,\ldots,d_{\Phi(x)}f_{k+l-1}\right).
\end{aligned}$$

Hence $\phi^2\circ\xi^2$ and $\phi^1\circ\xi^1$ are Φ-related, and the computation of Equations (8.2) and (8.3) shows that

$$\overline{\phi^1\circ\xi^1}(f_1\circ\Phi,\ldots,f_{k+l-1}\circ\Phi)=\overline{\phi^2\circ\xi^2}(f_1,\ldots,f_{k+l-1})\circ\Phi.$$

Similarly,

$$\overline{\xi^1\circ\phi^1}(f_1\circ\Phi,\ldots,f_{k+l-1}\circ\Phi)=\overline{\xi^2\circ\phi^2}(f_1,\ldots,f_{k+l-1})\circ\Phi.$$

At last,

$$\begin{aligned}
&[\phi^2,\xi^2]_{\Phi(x)}\left(d_{\Phi(x)}f_1,\ldots,d_{\Phi(x)}f_{k+l-1}\right)\\
&=\left(\phi^2\circ\xi^2\right)_{\Phi(x)}\left(d_{\Phi(x)}f_1,\ldots,d_{\Phi(x)}f_{k+l-1}\right)-(-1)^{(k-1)(l-1)}\left(\xi^2\circ\phi^2\right)_{\Phi(x)}\left(d_{\Phi(x)}f_1,\ldots,d_{\Phi(x)}f_{k+l-1}\right)\\
&=\overline{\phi^2\circ\xi^2}(f_1,\ldots,f_{k+l-1})(\Phi(x))-(-1)^{(k-1)(l-1)}\overline{\xi^2\circ\phi^2}(f_1,\ldots,f_{k+l-1})(\Phi(x))\\
&=\overline{\phi^1\circ\xi^1}(f_1\circ\Phi,\ldots,f_{k+l-1}\circ\Phi)(x)-(-1)^{(k-1)(l-1)}\overline{\xi^1\circ\phi^1}(f_1\circ\Phi,\ldots,f_{k+l-1}\circ\Phi)(x)\\
&=\left(\phi^1\circ\xi^1\right)_x\left(d_x(f_1\circ\Phi),\ldots,d_x(f_{k+l-1}\circ\Phi)\right)-(-1)^{(k-1)(l-1)}\left(\xi^1\circ\phi^1\right)_x\left(d_x(f_1\circ\Phi),\ldots,d_x(f_{k+l-1}\circ\Phi)\right)\\
&=\left(\phi^1\circ\xi^1\right)_x\left(d_{\Phi(x)}f_1\circ d_x\Phi,\ldots,d_{\Phi(x)}f_{k+l-1}\circ d_x\Phi\right)\\
&-(-1)^{(k-1)(l-1)}\left(\xi^1\circ\phi^1\right)_x\left(d_{\Phi(x)}f_1\circ d_x\Phi,\ldots,d_{\Phi(x)}f_{k+l-1}\circ d_x\Phi\right)\\
&=\left[\phi^1,\xi^1\right]_x\left(d_{\Phi(x)}f_1\circ d_x\Phi,\ldots,d_{\Phi(x)}f_{k+l-1}\circ d_x\Phi\right)\\
&=\left((d_x\Phi)\left[\phi^1,\xi^1\right]_x\right)\left(d_{\Phi(x)}f_1,\ldots,d_{\Phi(x)}f_{k+l-1}\right),
\end{aligned}$$

which finishes the proof. □

[1] Recall that for multivector fields $\zeta\in\mathfrak{X}^k(M),\nu\in\mathfrak{X}^l(M)$, we defined $\zeta\circ\nu$ in Definition 2.2.6.

8.4 Hadamard's Lemma

The proofs below are by [Vor].

Lemma 8.4.1 (Hadamard's Lemma)**.** *For any smooth function $f \in C^\infty(\mathbb{R}^n)$ and any $x_0 \in \mathbb{R}^n$, there is an expansion*

$$f(x) = f(x_0) + \sum_{i=1}^{n} (x^i - x_0^i) g_i(x),$$

where $g_i \in C^\infty(\mathbb{R}^n)$ are smooth functions.[2]

Proof. Holding x fixed, put $h(t) = f\left(x_0 + t(x - x_0)\right)$. Using the fundamental theorem of calculus and the chain rule, we get

$$\begin{aligned} f(x) - f(x_0) &= \int_0^1 h'(t)dt = \int_0^1 \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}\left(x_0 + t(x - x_0)\right)(x^i - x_0^i)dt \\ &= \sum_{i=1}^{n} (x^i - x_0^i) \int_0^1 \frac{\partial f}{\partial x^i}\left(x_0 + t(x - x_0)\right)dt. \end{aligned}$$

The lemma follows by putting $g_i(x) := \int_0^1 \frac{\partial f}{\partial x^i}\left(x_0 + t(x - x_0)\right)dt$. □

Corollary 8.4.2. *For any smooth function $f \in C^\infty(\mathbb{R}^n)$ and any $x_0 \in \mathbb{R}^n$, there is an expansion*

$$f(x) = f(x_0) + \sum_{i=1}^{n} (x^i - x_0^i) \frac{\partial f}{\partial x^i}(x_0) + \sum_{i,j=1}^{n} (x^i - x_0^i)(x^j - x_0^j) g_{ij}(x),$$

where $g_{ij} \in C^\infty(\mathbb{R}^n)$ are smooth functions.

Proof. Hadamard's Lemma gives that $f(x) = f(x_0) + \sum_{i=1}^{n}(x^i - x_0^i)g_i(x)$. Applying Hadamard's Lemma once more on the g_i, we obtain

$$f(x) = f(x_0) + \sum_{i=1}^{n} (x^i - x_0^i) a_i + \sum_{i,j=1}^{n} (x^i - x_0^i)(x^j - x_0^j) g_{ij}(x),$$

where $a_i \in \mathbb{R}$ are numbers and $g_{ij} \in C^\infty(\mathbb{R}^n)$ are functions. Applying the partial derivative $\frac{\partial}{\partial x^i}$ at x_0 on both sides yields $a_i = \frac{\partial f}{\partial x^i}(x_0)$. □

8.5 Adapted distance functions

This section complements Lemma 4.1.16 and justifies some of the claims made there. We show that any vector bundle admits a smooth metric, and we explicitly construct an adapted distance function λ as mentioned in Lemma 4.1.16.

Lemma 8.5.1. *Let V be a vector space with inner products $g_1, \ldots, g_n$. Then a positive linear combination of $g_1, \ldots, g_n$ is still an inner product on V.*

[2]We denote the coordinates $x = (x^1, \ldots, x^n)$ by upper indices to avoid double lower indices.

Proof. Choose $a_1, \dots, a_n \in \mathbb{R}_0^+$ and consider $g := \sum_{i=1}^n a_i g_i$. It is clear that g is bilinear and symmetric, since the g_i are. We check that g is positive-definite. Using that the g_i are positive-definite and that the a_i are positive, we have

$$\left(\sum_{i=1}^n a_i g_i\right)(v,v) = 0 \Leftrightarrow \sum_{i=1}^n a_i g_i(v,v) = 0 \Leftrightarrow \ \forall\ i: \ a_i g_i(v,v) = 0 \Leftrightarrow \ \forall\ i: \ g_i(v,v) = 0 \Leftrightarrow v = 0.$$

□

Lemma 8.5.2. *A vector bundle $\Pi : E \to M$ has a metric.*

Proof. Take an open cover $\{U_i\}$ of M that gives local trivializations $\phi_i : \Pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ of E. Then the standard inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ induces a metric g_i on the trivial bundles $\Pi^{-1}(U_i)$ by

$$g_i\big(\phi_i^{-1}(x,v), \phi_i^{-1}(x,w)\big) = \langle v, w \rangle.$$

Let $\{\psi_i : U_i \to [0,1]\}$ be a partition of unity subordinate to the cover $\{U_i\}$. Define $g := \sum_i \psi_i g_i$. So for e, e' with $\Pi(e) = \Pi(e') = x$, we have

$$g(e,e') = \sum_i \psi_i(x) g_i(e,e').$$

This is a metric on E by Lemma 8.5.1 above. □

Let (M, Z) be a b-manifold. We now construct an adapted distance function λ as mentioned in Lemma 4.1.16. Fix a tubular neighborhood $E \subset NZ$ of Z in the normal bundle, and let $p : E \to Z$ be the projection. By the above, we can take a metric g on E, and we have correspondingly a continuous distance function

$$\| \cdot \| : E \to \mathbb{R}^+ : x \mapsto \sqrt{g(x,x)}.$$

Define subsets

$$K := \{x \in E : \ \|x\| \leq 1/2\} = \| \cdot \|^{-1}([0, 1/2])$$

and

$$U := \{x \in E : \ \|x\| < 1\} = \| \cdot \|^{-1}([0,1)).$$

Then K is closed and U is open in E. Recall the smooth Urysohn lemma [Muk, Lemma 2.1.17]:

Lemma 8.5.3 (Smooth Urysohn Lemma)**.** *If $K \subset U \subset E$, where K is closed and U is open, then there is a smooth function $f : E \to \mathbb{R}$ such that $0 \leq f \leq 1$, $f|_K \equiv 1$ and supp(f) $\subset U$.*

Gluing such f with the zero function, we obtain $f : M \to \mathbb{R}$ such that $f|_K \equiv 1$ and $\mathrm{supp}(f) \subset U$. We now define $\lambda : M \to \mathbb{R}$ by $\lambda = f\| \cdot \| + (1 - f)$. Then it is clear that $\lambda(x) = \|x\|$ for $x \in E$ with $\|x\| \leq 1/2$, and $\lambda \equiv 1$ on $M \setminus \{x \in E : \ \|x\| < 1\}$. Restricting λ to $M \setminus Z$ gives a function as required in Lemma 4.1.16. We call this a distance function adapted to E.

8.6 A Poisson version of Cartan's magic formula

We present a proof of the following lemma, which we used in Theorem 4.3.8. It is stated without proof in [MO].

Lemma 8.6.1. *Let $\Pi \in \Gamma(\wedge^2 TM)$ be a Poisson bivector on M, and $\beta \in \Omega^1(M)$ a closed one-form. For a multivector field $\xi \in \mathfrak{X}^k(M)$, we have*

$$\iota_\beta(d_\Pi(\xi)) + d_\Pi(\iota_\beta(\xi)) = \mathcal{L}_{\Pi^\sharp(\beta)}\xi. \tag{8.4}$$

Proof. We check that (8.4) holds in local coordinates, and we proceed by induction on the degree of the multivector field. First consider $f \in \mathfrak{X}^0(M) = C^\infty(M)$. We write $\beta = \sum_i g_i dx_i$ and

$$\Pi = \sum_{i<j} \Pi_{i,j} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

Then

$$\begin{aligned}\Pi^\sharp(\beta) = \iota_\beta \left(\sum_{i<j} \Pi_{i,j} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right) &= \sum_{i<j} \Pi_{i,j} \iota_\beta \left(\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right) = \sum_{i<j} \Pi_{i,j} \left(g_i \frac{\partial}{\partial x_j} - g_j \frac{\partial}{\partial x_i} \right) \\ &= \sum_{i<j} \Pi_{i,j} g_i \frac{\partial}{\partial x_j} - \sum_{j<i} \Pi_{j,i} g_i \frac{\partial}{\partial x_j} = \sum_{i<j} \Pi_{i,j} g_i \frac{\partial}{\partial x_j} + \sum_{j<i} \Pi_{i,j} g_i \frac{\partial}{\partial x_j} = \sum_{i,j} \Pi_{i,j} g_i \frac{\partial}{\partial x_j}.\end{aligned}$$

Hence

$$\mathcal{L}_{\Pi^\sharp(\beta)} f = df(\Pi^\sharp(\beta)) = \sum_{i,j} \Pi_{i,j} g_i \frac{\partial f}{\partial x_j}.$$

Now $d_\Pi(\iota_\beta(f)) = d_\Pi(0) = 0$ and

$$\begin{aligned}\iota_\beta(d_\Pi(f)) = \iota_\beta([\Pi, f]) = \iota_\beta(-\iota_{df}\Pi) &= -\iota_\beta \left(\sum_{i<j} \Pi_{i,j} \left(\frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i} \right) \right) \\ &= \sum_{i<j} \Pi_{i,j} \frac{\partial f}{\partial x_j} g_i - \sum_{i<j} \Pi_{i,j} \frac{\partial f}{\partial x_i} g_j = \sum_{i<j} \Pi_{i,j} g_i \frac{\partial f}{\partial x_j} - \sum_{j<i} \Pi_{j,i} g_i \frac{\partial f}{\partial x_j} \\ &= \sum_{i<j} \Pi_{i,j} g_i \frac{\partial f}{\partial x_j} + \sum_{j<i} \Pi_{i,j} g_i \frac{\partial f}{\partial x_j} = \sum_{i,j} \Pi_{i,j} g_i \frac{\partial f}{\partial x_j}.\end{aligned}$$

So (8.4) holds on functions. Now let $X = \sum_i f_i \frac{\partial}{\partial x_i} \in \mathfrak{X}^1(M)$ and Π, β given in coordinates as before. By the above, we know that

$$\Pi^\sharp(\beta) = \sum_{i,j} \Pi_{i,j} g_i \frac{\partial}{\partial x_j} = \sum_j \left(\sum_i \Pi_{i,j} g_i \right) \frac{\partial}{\partial x_j}.$$

Then

$$\begin{aligned}\mathcal{L}_{\Pi^\sharp(\beta)}(X) &= \left[\sum_j \left(\sum_i \Pi_{i,j} g_i \right) \frac{\partial}{\partial x_j}, \sum_k f_k \frac{\partial}{\partial x_k} \right] \\ &= \sum_{j,k} \left[\left(\sum_i \Pi_{i,j} g_i \frac{\partial f_k}{\partial x_j} \right) \frac{\partial}{\partial x_k} - f_j \frac{\partial(\sum_i \Pi_{i,k} g_i)}{\partial x_j} \frac{\partial}{\partial x_k} \right]\end{aligned}$$

$$= \sum_{i,j,k} \Pi_{i,j} g_i \frac{\partial f_k}{\partial x_j} \frac{\partial}{\partial x_k} - \sum_{j,k} f_j \sum_i \left(\frac{\partial \Pi_{i,k}}{\partial x_j} g_i + \Pi_{i,k} \frac{\partial g_i}{\partial x_j} \right) \frac{\partial}{\partial x_k}$$

$$= \sum_{i,j,k} \Pi_{i,j} g_i \frac{\partial f_k}{\partial x_j} \frac{\partial}{\partial x_k} - \sum_{i,j,k} f_j \frac{\partial \Pi_{i,k}}{\partial x_j} g_i \frac{\partial}{\partial x_k} - \sum_{i,j,k} f_j \Pi_{i,k} \frac{\partial g_i}{\partial x_j} \frac{\partial}{\partial x_k}.$$

On the other hand, since $\iota_\beta(X) = \sum_k f_k g_k$, we get

$$d_\Pi(\iota_\beta(X)) = \left[\Pi, \sum_k f_k g_k \right] = -\iota_{d\left(\sum_k f_k g_k\right)} \Pi$$

$$= -\sum_{i<j} \Pi_{i,j} \left(\frac{\partial \left(\sum_k f_k g_k\right)}{\partial x_i} \frac{\partial}{\partial x_j} - \frac{\partial \left(\sum_k f_k g_k\right)}{\partial x_j} \frac{\partial}{\partial x_i} \right)$$

$$= -\sum_{i<j} \Pi_{i,j} \sum_k \left(\frac{\partial f_k}{\partial x_i} g_k + f_k \frac{\partial g_k}{\partial x_i} \right) \frac{\partial}{\partial x_j} + \sum_{i<j} \Pi_{i,j} \sum_k \left(\frac{\partial f_k}{\partial x_j} g_k + f_k \frac{\partial g_k}{\partial x_j} \right) \frac{\partial}{\partial x_i}$$

$$= \sum_{i<j} \sum_k \Pi_{i,j} \frac{\partial f_k}{\partial x_j} g_k \frac{\partial}{\partial x_i} + \sum_{i<j} \sum_k \Pi_{i,j} f_k \frac{\partial g_k}{\partial x_j} \frac{\partial}{\partial x_i}$$

$$- \sum_{i<j} \sum_k \Pi_{i,j} \frac{\partial f_k}{\partial x_i} g_k \frac{\partial}{\partial x_j} - \sum_{i<j} \sum_k \Pi_{i,j} f_k \frac{\partial g_k}{\partial x_i} \frac{\partial}{\partial x_j}$$

$$= \sum_{i,j,k} \Pi_{i,j} \frac{\partial f_k}{\partial x_j} g_k \frac{\partial}{\partial x_i} + \sum_{i,j,k} \Pi_{i,j} f_k \frac{\partial g_k}{\partial x_j} \frac{\partial}{\partial x_i}.$$

Next, we have

$$d_\Pi(X) = \left[\Pi, \sum_k f_k \frac{\partial}{\partial x_k} \right] = -\left[\sum_k f_k \frac{\partial}{\partial x_k}, \sum_{i<j} \Pi_{i,j} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right]$$

$$= -\pounds_{\sum_k f_k \frac{\partial}{\partial x_k}} \left(\sum_{i<j} \Pi_{i,j} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \right)$$

$$= -\sum_{i<j} \Bigg[\left(\pounds_{\sum_k f_k \frac{\partial}{\partial x_k}} \Pi_{i,j} \right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \Pi_{i,j} \left(\pounds_{\sum_k f_k \frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i} \right) \wedge \frac{\partial}{\partial x_j}$$

$$+ \Pi_{i,j} \frac{\partial}{\partial x_i} \wedge \left(\pounds_{\sum_k f_k \frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_j} \right) \Bigg]$$

$$= -\sum_{i<j} \sum_k f_k \frac{\partial \Pi_{i,j}}{\partial x_k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \sum_{i<j} \Pi_{i,j} \left(\pounds_{\frac{\partial}{\partial x_i}} \left(\sum_k f_k \frac{\partial}{\partial x_k} \right) \right) \wedge \frac{\partial}{\partial x_j}$$

$$+ \sum_{i<j} \Pi_{i,j} \frac{\partial}{\partial x_i} \wedge \left(\pounds_{\frac{\partial}{\partial x_j}} \left(\sum_k f_k \frac{\partial}{\partial x_k} \right) \right)$$

$$= -\sum_{i<j} \sum_k f_k \frac{\partial \Pi_{i,j}}{\partial x_k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \sum_{i<j} \Pi_{i,j} \sum_k \frac{\partial f_k}{\partial x_i} \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_j}$$

$$+ \sum_{i<j} \Pi_{i,j} \frac{\partial}{\partial x_i} \wedge \left(\sum_k \frac{\partial f_k}{\partial x_j} \frac{\partial}{\partial x_k} \right)$$

$$= -\sum_{i<j} \sum_k f_k \frac{\partial \Pi_{i,j}}{\partial x_k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \sum_{i<j} \sum_k \Pi_{i,j} \frac{\partial f_k}{\partial x_i} \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial x_j}$$

$$+\sum_{i<j}\sum_k \Pi_{i,j}\frac{\partial f_k}{\partial x_j}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_k}.$$

Re-indexing, we have

$$\sum_{i<j}\sum_k \Pi_{i,j}\frac{\partial f_k}{\partial x_j}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_k}=\sum_{j<i}\sum_k \Pi_{j,i}\frac{\partial f_k}{\partial x_i}\frac{\partial}{\partial x_j}\wedge\frac{\partial}{\partial x_k}=\sum_{j<i}\sum_k \Pi_{i,j}\frac{\partial f_k}{\partial x_i}\frac{\partial}{\partial x_k}\wedge\frac{\partial}{\partial x_j},$$

so that

$$d_\Pi(X)=-\sum_{i<j}\sum_k f_k\frac{\partial \Pi_{i,j}}{\partial x_k}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}+\sum_{i,j,k}\Pi_{i,j}\frac{\partial f_k}{\partial x_i}\frac{\partial}{\partial x_k}\wedge\frac{\partial}{\partial x_j}.$$

Hence

$$\begin{aligned}
\iota_\beta(d_\Pi(X))&=-\sum_{i<j}\sum_k f_k\frac{\partial \Pi_{i,j}}{\partial x_k}\left(g_i\frac{\partial}{\partial x_j}-g_j\frac{\partial}{\partial x_i}\right)+\sum_{i,j,k}\Pi_{i,j}\frac{\partial f_k}{\partial x_i}\left(g_k\frac{\partial}{\partial x_j}-g_j\frac{\partial}{\partial x_k}\right)\\
&=-\sum_{i<j}\sum_k f_k\frac{\partial \Pi_{i,j}}{\partial x_k}g_i\frac{\partial}{\partial x_j}+\sum_{i<j}\sum_k f_k\frac{\partial \Pi_{i,j}}{\partial x_k}g_j\frac{\partial}{\partial x_i}\\
&\quad+\sum_{i,j,k}\Pi_{i,j}\frac{\partial f_k}{\partial x_i}g_k\frac{\partial}{\partial x_j}-\sum_{i,j,k}\Pi_{i,j}\frac{\partial f_k}{\partial x_i}g_j\frac{\partial}{\partial x_k}\\
&=\sum_{i,j,k}f_k\frac{\partial \Pi_{i,j}}{\partial x_k}g_j\frac{\partial}{\partial x_i}+\sum_{i,j,k}\Pi_{i,j}\frac{\partial f_k}{\partial x_i}g_k\frac{\partial}{\partial x_j}-\sum_{i,j,k}\Pi_{i,j}\frac{\partial f_k}{\partial x_i}g_j\frac{\partial}{\partial x_k}.
\end{aligned}$$

So

$$\begin{aligned}
\iota_\beta(d_\Pi(X))+d_\Pi(\iota_\beta(X))&=\sum_{i,j,k}\Pi_{i,j}\frac{\partial f_k}{\partial x_j}g_k\frac{\partial}{\partial x_i}+\sum_{i,j,k}\Pi_{i,j}f_k\frac{\partial g_k}{\partial x_j}\frac{\partial}{\partial x_i}+\sum_{i,j,k}f_k\frac{\partial \Pi_{i,j}}{\partial x_k}g_j\frac{\partial}{\partial x_i}\\
&\quad+\sum_{i,j,k}\Pi_{i,j}\frac{\partial f_k}{\partial x_i}g_k\frac{\partial}{\partial x_j}-\sum_{i,j,k}\Pi_{i,j}\frac{\partial f_k}{\partial x_i}g_j\frac{\partial}{\partial x_k}\\
&=\sum_{i,j,k}\Pi_{i,j}f_k\frac{\partial g_k}{\partial x_j}\frac{\partial}{\partial x_i}+\sum_{i,j,k}f_k\frac{\partial \Pi_{i,j}}{\partial x_k}g_j\frac{\partial}{\partial x_i}-\sum_{i,j,k}\Pi_{i,j}\frac{\partial f_k}{\partial x_i}g_j\frac{\partial}{\partial x_k}.
\end{aligned}$$

We now inspect

$$\mathcal{L}_{\Pi^\sharp(\beta)}(X)=\sum_{i,j,k}\Pi_{i,j}g_i\frac{\partial f_k}{\partial x_j}\frac{\partial}{\partial x_k}-\sum_{i,j,k}f_j\frac{\partial \Pi_{i,k}}{\partial x_j}g_i\frac{\partial}{\partial x_k}-\sum_{i,j,k}f_j\Pi_{i,k}\frac{\partial g_i}{\partial x_j}\frac{\partial}{\partial x_k}.$$

Re-indexing gives

$$\sum_{i,j,k}f_k\frac{\partial \Pi_{i,j}}{\partial x_k}g_j\frac{\partial}{\partial x_i}=\sum_{i,j,k}f_i\frac{\partial \Pi_{k,j}}{\partial x_i}g_j\frac{\partial}{\partial x_k}=\sum_{i,j,k}f_j\frac{\partial \Pi_{k,i}}{\partial x_j}g_i\frac{\partial}{\partial x_k}=-\sum_{i,j,k}f_j\frac{\partial \Pi_{i,k}}{\partial x_j}g_i\frac{\partial}{\partial x_k}$$

and

$$-\sum_{i,j,k}\Pi_{i,j}\frac{\partial f_k}{\partial x_i}g_j\frac{\partial}{\partial x_k}=-\sum_{i,j,k}\Pi_{j,i}g_i\frac{\partial f_k}{\partial x_j}\frac{\partial}{\partial x_k}=\sum_{i,j,k}\Pi_{i,j}g_i\frac{\partial f_k}{\partial x_j}\frac{\partial}{\partial x_k}.$$

Since β is closed, we have

$$0=d\beta=d\left(\sum_i g_i dx_i\right)=\sum_i dg_i\wedge dx_i=\sum_i\sum_j\frac{\partial g_i}{\partial x_j}dx_j\wedge dx_i$$

$$
\begin{aligned}
&= \sum_{i<j} \frac{\partial g_i}{\partial x_j} dx_j \wedge dx_i + \sum_{j<i} \frac{\partial g_i}{\partial x_j} dx_j \wedge dx_i \\
&= \sum_{j<i} \frac{\partial g_j}{\partial x_i} dx_i \wedge dx_j + \sum_{j<i} \frac{\partial g_i}{\partial x_j} dx_j \wedge dx_i \\
&= \sum_{j<i} \left(\frac{\partial g_i}{\partial x_j} - \frac{\partial g_j}{\partial x_i} \right) dx_j \wedge dx_i.
\end{aligned}
$$

Hence

$$
\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i} \qquad \text{for all } i, j.
$$

This then implies

$$
\begin{aligned}
\sum_{i,j,k} \Pi_{i,j} f_k \frac{\partial g_k}{\partial x_j} \frac{\partial}{\partial x_i} &= \sum_{i,j,k} \Pi_{k,j} f_i \frac{\partial g_i}{\partial x_j} \frac{\partial}{\partial x_k} = \sum_{i,j,k} \Pi_{k,i} f_j \frac{\partial g_j}{\partial x_i} \frac{\partial}{\partial x_k} \\
&= -\sum_{i,j,k} \Pi_{i,k} f_j \frac{\partial g_j}{\partial x_i} \frac{\partial}{\partial x_k} = -\sum_{i,j,k} \Pi_{i,k} f_j \frac{\partial g_i}{\partial x_j} \frac{\partial}{\partial x_k}
\end{aligned}
$$

So we conclude that

$$
\pounds_{\Pi^\sharp(\beta)}(X) = \iota_\beta(d_\Pi(X)) + d_\Pi(\iota_\beta(X)).
$$

Now assume (8.4) holds for $(k-1)$-vector fields and let $\xi \in \mathfrak{X}^k(M)$. Since both sides of (8.4) are $\mathbb{R}$-linear, we may assume that

$$
\xi = f \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \wedge \cdots \wedge \frac{\partial}{\partial x_k}.
$$

We write this as

$$
\xi = \frac{\partial}{\partial x_1} \wedge \xi_1 \qquad \text{with } \xi_1 = f \frac{\partial}{\partial x_2} \wedge \cdots \wedge \frac{\partial}{\partial x_k} \in \mathfrak{X}^{k-1}(M).
$$

Then

$$
\begin{aligned}
d_\Pi(\iota_\beta(\xi)) + \iota_\beta(d_\Pi(\xi)) &= d_\Pi\left(\iota_\beta\left(\frac{\partial}{\partial x_1} \wedge \xi_1\right)\right) + \iota_\beta\left(d_\Pi\left(\frac{\partial}{\partial x_1} \wedge \xi_1\right)\right) \\
&= d_\Pi\left(\iota_\beta\left(\frac{\partial}{\partial x_1}\right)\xi_1 - \frac{\partial}{\partial x_1} \wedge \iota_\beta(\xi_1)\right) + \iota_\beta\left(d_\Pi\left(\frac{\partial}{\partial x_1}\right) \wedge \xi_1 - \frac{\partial}{\partial x_1} \wedge d_\Pi(\xi_1)\right) \\
&= d_\Pi\left(\iota_\beta\left(\frac{\partial}{\partial x_1}\right)\right) \wedge \xi_1 + \iota_\beta\left(\frac{\partial}{\partial x_1}\right) d_\Pi(\xi_1) - d_\Pi\left(\frac{\partial}{\partial x_1}\right) \wedge \iota_\beta(\xi_1) \\
&\quad + \frac{\partial}{\partial x_1} \wedge d_\Pi(\iota_\beta(\xi_1)) + \iota_\beta\left(d_\Pi\left(\frac{\partial}{\partial x_1}\right)\right) \wedge \xi_1 + d_\Pi\left(\frac{\partial}{\partial x_1}\right) \wedge \iota_\beta(\xi_1) \\
&\quad - \iota_\beta\left(\frac{\partial}{\partial x_1}\right) d_\Pi(\xi_1) + \frac{\partial}{\partial x_1} \wedge \iota_\beta\left(d_\Pi(\xi_1)\right) \\
&= \left[d_\Pi\left(\iota_\beta\left(\frac{\partial}{\partial x_1}\right)\right) + \iota_\beta\left(d_\Pi\left(\frac{\partial}{\partial x_1}\right)\right)\right] \wedge \xi_1 \\
&\quad + \frac{\partial}{\partial x_1} \wedge \left[d_\Pi(\iota_\beta(\xi_1)) + \iota_\beta\left(d_\Pi(\xi_1)\right)\right] \\
&= \left(\pounds_{\Pi^\sharp(\beta)} \frac{\partial}{\partial x_1}\right) \wedge \xi_1 + \frac{\partial}{\partial x_1} \wedge \left(\pounds_{\Pi^\sharp(\beta)} \xi_1\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \pounds_{\Pi^\sharp(\beta)} \left(\frac{\partial}{\partial x_1} \wedge \xi_1 \right) \\
&= \pounds_{\Pi^\sharp(\beta)} \xi.
\end{aligned}
$$

□

8.7 On modular vector fields

In Section 6.2, we used the following result. Its proof goes after [LPV, Proposition 4.14].

Theorem 8.7.1. *Let M^{2n+1} be an orientable manifold with a corank-one Poisson structure Π. Then the modular vector field X_Π^Ω is tangent to the symplectic leaves of M, for any choice of volume form Ω.*

Proof. Choose $p \in M$ and let $(U, q_1, \dots, q_n, p_1, \dots, p_n, z)$ be splitting coordinates around p, so that

$$
\Pi|_U = \sum_{i=1}^{n} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}. \tag{8.5}
$$

We consider the volume form Λ on U, defined by

$$
\Lambda := dq_1 \wedge dp_1 \wedge \cdots \wedge dq_n \wedge dp_n \wedge dz,
$$

and for each $k \in \{1, \dots, n\}$ we denote

$$
\lambda_k := \left[\bigwedge_{i \neq k} (dq_i \wedge dp_i) \right] \wedge dz.
$$

Then for $k \in \{1, \dots, n\}$ and for $f \in C^\infty(M)$ we have:

$$
\begin{aligned}
\pounds_{X_f}(dq_k \wedge dp_k) &= \pounds_{X_f}(dq_k) \wedge dp_k + dq_k \wedge \pounds_{X_f}(dp_k) \\
&= d\left(\pounds_{X_f} q_k\right) \wedge dp_k + dq_k \wedge d\left(\pounds_{X_f} p_k\right) \\
&= d\left(X_f(q_k)\right) \wedge dp_k + dq_k \wedge d\left(X_f(p_k)\right) \\
&= -d\left(\frac{\partial f}{\partial p_k}\right) \wedge dp_k + dq_k \wedge d\left(\frac{\partial f}{\partial q_k}\right).
\end{aligned}
$$

Here we used Lemma 2.7.4 to find that

$$
X_f(q_k) = -\frac{\partial f}{\partial p_k} \quad \text{and} \quad X_f(p_k) = \frac{\partial f}{\partial q_k}.
$$

Therefore, for all $k \in \{1, \dots, n\}$, we find that $\lambda_k \wedge \pounds_{X_f}(dq_k \wedge dp_k)$ equals

$$
\begin{aligned}
&\lambda_k \wedge \left(dq_k \wedge d\left(\frac{\partial f}{\partial q_k}\right) + dp_k \wedge d\left(\frac{\partial f}{\partial p_k}\right) \right) \\
&\quad = dq_1 \wedge dp_1 \wedge \cdots \wedge dq_{k-1} \wedge dp_{k-1} \wedge dq_{+1} \wedge dp_{k+1} \wedge \cdots \wedge dq_n \wedge dp_n \wedge dz \wedge \\
&\quad \left[dq_k \wedge \left(\sum_{i=1}^{n} \frac{\partial^2 f}{\partial q_i \partial q_k} dq_i + \sum_{j=1}^{n} \frac{\partial^2 f}{\partial p_j \partial q_k} dp_j + \frac{\partial^2 f}{\partial z \partial q_k} dz \right) \right. \\
&\quad \left. + dp_k \wedge \left(\sum_{i=1}^{n} \frac{\partial^2 f}{\partial q_i \partial p_k} dq_i + \sum_{j=1}^{n} \frac{\partial^2 f}{\partial p_j \partial p_k} dp_j + \frac{\partial^2 f}{\partial z \partial p_k} dz \right) \right]
\end{aligned}
$$

$$= dq_1 \wedge dp_1 \wedge \cdots \wedge dq_{k-1} \wedge dp_{k-1} \wedge dq_{k+1} \wedge dp_{k+1} \wedge \cdots \wedge dq_n \wedge dp_n \wedge dz \wedge dq_k \wedge \frac{\partial^2 f}{\partial p_k \partial q_k} dp_k$$
$$+ dq_1 \wedge dp_1 \wedge \cdots \wedge dq_{k-1} \wedge dp_{k-1} \wedge dq_{k+1} \wedge dp_{k+1} \wedge \cdots \wedge dq_n \wedge dp_n \wedge dz \wedge dp_k \wedge \frac{\partial^2 f}{\partial q_k \partial p_k} dq_k$$
$$= 0.$$

It then follows that

$$\begin{aligned}
\pounds_{X_f} \Lambda &= \pounds_{X_f}(dq_1 \wedge dp_1 \wedge \cdots \wedge dq_n \wedge dp_n \wedge dz) \\
&= \sum_{k=1}^{n} \pounds_{X_f}(dq_k \wedge dp_k) \wedge \lambda_k + dq_1 \wedge dp_1 \wedge \cdots \wedge dq_k \wedge dp_k \wedge \pounds_{X_f}(dz) \\
&= dq_1 \wedge dp_1 \wedge \cdots \wedge dq_k \wedge dp_k \wedge \pounds_{X_f}(dz) \\
&= 0,
\end{aligned}$$

using Lemma 2.7.4 and (8.5) to see that $\pounds_{X_f}(dz) = 0$. It follows that for the volume form Λ, the modular vector field is zero on U. Hence for an arbitrary volume form on U, the modular vector field is Hamiltonian on U, in particular tangent to the symplectic leaves. □

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